British Journal of Mathematics & Computer Science
10(4): 1-11, 2015, Article no.BJMCS.18725
ISSN: 2231-0851

SCIENCEDOMAIN *international*
www.sciencedomain.org

SCIENCEDOMAIN

Nonlinear Impulsive Fractional Integro Differential Equations

Hamdy M. Ahmed[1*], A. A. M. Hassan[2] and A. S. Ghanem[1]

[1] *Higher Institute of Engineering, El-Shorouk Academy, P.O. 3 El-Shorouk City, Cairo, Egypt.*
[2] *Department of Mathematics, Faculty of Science, Zagazig University, Egypt.*

Article Information

DOI: 10.9734/BJMCS/2015/18725
Editor(s):
(1) Metin Baarir, Department of Mathematics, Sakarya University, Turkey.
Reviewers:
(1) Anonymous, India.
(2) Mark McKibben, West Chester University, USA.
(3) Anonymous, Northwest Normal University, China.
Complete Peer review History: http://sciencedomain.org/review-history/10313

Original Research Article

Received: 07 May 2015
Accepted: 18 June 2015
Published: 24 July 2015

Abstract

By using fixed point theorem we studied the mild solution of fractional integro- differential equations with non-local and impulsive conditions, also we studied the sufficient conditions of controllability for this system.

Keywords: Fractional calculus; nonlocal conditions; impulsive conditions; Integro-differential systems; mild solutions; Semigroup theory; probability density function; controllability.

2010 Mathematics Subject Classification: 53C25; 83C05; 57N16.

1 Introduction

In recent years, considerable interest in fractional calculus has been stimulated by the applications it finds in numerical analysis and different areas of applied sciences like physics and engineering [1-7].Fractal phenomena often can be centered in the field of linear viscoelasticity. One of the major advantages of fractional calculus is that it can be considered as a super set of integer-order calculus. Thus, fractional calculus has the potential to accomplish what integer-order calculus cannot. The interested reader in this topic can consult the excellent books [8,9]. In particular the non-local problems for impulsive fractional differential equations have been attractive to many researchers the advantage of impulsive fractional differential equations is that they can describe the model

**Corresponding author: E-mail: hamdy$_1$7eg@yahoo.com*

which at certain moments change their state rapidly and which can't be modeled by the classical differential equations.

Controllability is an important property of a control system, and the controllability property plays a crucial role in many control problems, such as stabilization of unstable systems by feedback, or optimal control.Controllability and observability are dual aspects of the same problem. Roughly, the concept of controllability denotes the ability to move a system around in its entire configuration space using only certain admissible manipulations. The exact definition varies slightly within the framework or the type of models applied. The following are examples of variations of controllability notions which have been introduced in the systems and control literature: State controllability, Output controllability, and Controllability in the behavioral framework (see [10-15]) The main purpose of this paper is to prove the existence of mild solutions and controllability for the following impulsive fractional integro- differential equation with nonlocal condition in a Banach space X:

$$^{c}D_0^q[x(t) - F(t, x(t), x(b_1(t)), ..., x(b_m(t))] = A[x(t) - F(t, x(t), x(b_1(t)), ..., x(b_m(t))]$$

$$+f(t, x(t), x(a_1(t)), ..., x(a_n(t)))+\int_0^t g(t, s, x(s), \psi(s)),\ t \in J = [0, T], t \neq t_k, k = 1, 2, 3......, m \quad (1.1)$$

$$x(0) + h(x(t_1), ..., x(t_p)) = x_0 \qquad (1.2)$$

$$\Delta x|_{t=t_k} = I_k(x(t_k^-)), k = 1, 2, 3, ...m \qquad (1.3)$$

The linear operator A generates an analytic semigroup $(T(t))_{t\geq 0}$

where, $(T(t))_{t\geq 0}$ is a compact analytic semigroup of uniformly bounded linear operators $T(t)$ on X, $(T(t))_{t\geq 0}$ is a compact analytic semigroup of uniformly bounded linear operators $(T(t))$ on X

$$\Delta x|_{t=t_k} = I_k(x(t_k^-))$$

where, $x(t_k^+)$ is the right limit of $x(t)$ at $(t = t_k)$, $x(t_k^-)$ is the left limit of $x(t)$ at $(t = t_k)$. F, G and g are given functions to be specified later and $^{c}D_0^q$ is Caputo fractional derivative of order $0 < q < 1$.

1.1 Preliminaries

Let X be a Banach space with norm $\|.\|$ and A :$D(A) \to X$ is the generator of a compact analytic semigroup of uniformly bounded linear operators $(T(t))$ on X.

there exist $M \geq 1$ such that $\|T(t)\| \leq M, t \geq 0$

We need some basic definitions and properties of the fractional calculus theory which are used in this paper

Definition 2.1 (see[16],[17]) The fractional integral of order q with the lower limit 0 for a function f is defined as:

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds,$$

$t > 0$, $q > 0$ where Γ is the gamma function.

Definition 2.2 (see[16],[17]) The Caputo derivative of order q with the lower limit 0 for a function f is defined as:

$$^{c}D_0^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}} ds,$$

$t > 0 \quad , 0 \leq n-1 < q < n$

Definition 2.3 a continuous solution $x(t)$of the integral equation

$$x(t) = S_q(t)[x_0 - h(x(t_1),...,x(t_p)) - F(0,x(0),x(b_1(0)),.....,x(b_m(0)))] + F(t,x(t),x(b_1(t)),...,x(b_m(t))$$

$$+ \int_0^t (t-s)^{q-1} T_q(t-s)[f(s,x(s),x(a_1(s)),......,x(a_n(s)) + \int_0^s g(s,\tau,x(\tau),\psi(\tau))d\tau]ds$$

$$+ \sum_{0<t_k<t} T_q(t-t_k) I_k(x(t_k^-))$$

is called mild solution of the problem (1.1)-(1.3) where

$$S_q(t)\ x = \int_0^\infty \xi_q(\theta) T(t^q\theta) x d\theta$$

$$T_q(t)\ x = q \int_0^\infty \theta \xi_q(\theta) T(t^q\theta) x d\theta$$

with ξ_q being a probability density function defined on $(0,\infty)$,that is
$\xi_q(\theta) \geq 0, \theta \in (0,\infty)$ and $\int_0^\infty \xi_q(\theta)d\theta = 1$
let$Y = C(J,X)$ and define the sets

$$X_r = \{x \in X : \|x\| \leq r\}, Y_r = \{y \in Y : \|y\| = sup_{t\in J}\|y(t)\| \leq r\}$$

where r ,positive constant,is defined by:

$$r = M[\|x_0\| + H + M_0L_1(k+1) + \frac{t^\alpha}{\Gamma(\alpha+1)}(M_2 + M_3) + \alpha\sum_{k=1}^m \lambda_k] + M_0L_1(k+1)$$

further we assume the following hypotheses:

$(H1) F : J \times X^{m+1} \to X$ is continuous function there exist constant $L_1 > 0$ such that:
$\|A^\beta F(t,x_0,x_1,......,x_m)\| \leq L_1(max\{\|x_i\| : i = 0,1,....,m\} + 1)$, holds for any $(t,x_0,x_1,......,x_m) \in J \times X^{m+1}$,$\|A^{-\beta}\| \leq M_0$,where β is constant
$(H2)$ the nonlinear operators$f : J \times X^{n+1} \to X, g : \triangle \times X \times X \to X$ and
$k : \triangle \times X \to X$ are continuous and there exist $M_2 > 0, M_3 > 0$ such that
we will show that $\phi(Y_0) = S = (\phi x) : x \in Y_0$ is an equicontinuous family of functions for$0 \leq t \leq s$ we have

$\|f(t,x(t),x(a_1(t)),...,x(a_n(t))\| \leq M_2$ for $t \in J$,$x \in X_r$,$\|g(t,s,x(s),y(s))\| \leq M_3$ for $(t,s) \in \triangle$,$x,y \in X_r$
$(H3)$ the operator $h : X^p \to X$ is continuous and there exist a constant $H > 0$ such that
$\|h(x(t_1),.....,x(t_p)\| \leq H$ for $x \in Y_r$
$h(\lambda x(t_1) + (1-\lambda)y(t_1),....,\lambda x(t_p) + (1-\lambda)y(t_p) = \lambda h(x(t_1),.....,x(t_p)$
$+ (1-\lambda)h(y(t_1),.....,y(t_p)$ for $x,y \in Y_r$
$(H4)$ the set $y(0) : y \in Y_r, y(0) = x_0 - h(y(t_1,.....,y(t_p))$ is precompact in X

lemma 2.1. . the operators $S_q(t)$ and $T_q(t)$ have the following properties :
(I) for any fixed $x \in X$,$\|S_q(t)x\| \leq M\|x\|$,$\|T_q(t)x\| \leq \frac{qM}{\Gamma(q+1)}\|x\|$
(II) $S_q(t), t \geq 0$ and $T_q(t), t \geq 0$ are strongly continuous
(III) For every $t > 0$,$S_q(t)$ and $T_q(t)$ are also compact operators if $T(t), t > 0$ is compact
$(H5)$ $I_k : X \to X$ is completely continues and their exist continuous non-decreasing functions

$L_k : R_+ \rightarrow R_+$ Such that for each $x \in X$.

$$\|I_k(x)\| \leq L_k(\|x\|), L_k(.) = \lambda_k \Gamma(q+1)$$

2 Existence of Mild Solutions

in this section we can prove the existence of mild solution (see[18-20])

Theorem 3.1.[16] Let hypotheses $(H1) - (H5)$ be satisfied then the system (1.1)-(1.3) has a mild solution on J.

proof.
For simplicity we rewrite that

$$(t, x(t), x(b_1(t), ..., x(b_m(t))) = (t, v(t))$$

and

$$(t, x(t), x(a_1(t), ..., x(a_n(t))) = (t, u(t))$$

we define the set Y_0 in $Y$by:

$Y_0 = \{x \in y : x(0) + h(x(t_1), ..., x(t_p) = x_0, \|x(t)\| \leq r \text{ for } 0 \leq t \leq T\}$
it is clear that Y_0 is a bounded closed convex subset of Y.

define a mapping $\phi : Y \rightarrow Y_0$ by:

$(\phi x)(t) = S_q(t)[x_0 - h(x(t_1), ..., x(t_p)) - F(0, v(0))] + F(t, v(t))$
$+ \int_0^t (t-s)^{q-1} T_q(t-s)[f(s, u(s) + \int_0^s g(s, \tau, x(\tau), \psi(\tau))d\tau]ds + \sum_{0<t_k<t} T_q(t-t_k) I_k(x(t_k^-))$
since , $\|(\phi x)(t)\| \leq \|S_q(t)x_0\| + \|S_q(t)h(x(t_1), ..., x(t_p))\| + \|S_q(t)F(0, v(0))\| + \|F(t, v(t))\| + \int_0^t (t-s)^{q-1} \|T_q(t-s)\| [\|f(s, u(s)\| + \int_0^s \|g(s, \tau, x(\tau), \psi(\tau))\| d\tau]ds + \sum_{0<t_k<t} \|T_q(t-t_k) I_k(x(t_k^-))\|$
$\|(\phi x)(t)\| \leq \|S_q(t)x_0\| + \|S_q(t)h(x(t_1), ..., x(t_p))\| + \|S_q(t)A^{-\beta}A^{\beta}F(0, v(0))\| + \|A^{-\beta}A^{\beta}F(t, v(t))\| + \int_0^t (t-s)^{q-1} \|T_q(t-s)\| [\|f(s, u(s)\| + \int_0^s \|g(s, \tau, x(\tau), \psi(\tau))\| d\tau]ds$
$+ \sum_{0<t_k<t} \|T_q(t-t_k)\| \|I_k(x(t_k^-))\|$
$\|(\phi x)(t)\| \leq M\|x_0\| + MH + MM_0L_1(k+1) + M_0L_1(k+1) + \frac{t^q}{\Gamma(q+1)} M(M_2 + M_3)$
$+ \frac{qM}{\Gamma(q+1)} \sum_{k=1}^m L_k$
$\|(\phi x)(t)\| \leq M[\|x_0\| + H + M_0L_1(k+1) + \frac{t^q}{\Gamma(q+1)}(M_2 + M_3) + q\sum_{k=1}^m \lambda_k]$
$+ M_0L_1(k+1) = r$

then $\|\phi\| \leq r$ this is mean that ϕ maps Y_0 into Y_0 further the continuity of ϕ fromY_0 into Y_0 follows from the fact that (f, g, k, F, h) are continuous.

moreover ϕ maps Y_0 into a precompact subset of Y_0
we prove that the set
$Y_0(t) = \{(\phi x)(t) : x \in Y_0\}$ is precompact in X , for $t = o$,the set$Y_0(0)$ is precompact in X this is from the condition $(H4)$

let $t > 0$ be fixed define , for $0 < \epsilon < t$,
$(\phi_\epsilon x)(t) = S_q(t)[x_0 - h(x(t_1), ..., x(t_p)) - F(0, v(0))] + F(t, v(t))$
$+ \int_0^{t-\epsilon} (t-s)^{q-1} T_q(t-s)[f(s, u(s) + \int_0^s g(s, \tau, x(\tau), \psi(\tau))d\tau]ds$
$+ \sum_{0<t_k<t} T_q(t-t_k) I_k(x(t_k^-))$

since $T(t)$ is compact analytic semigroup of uniformly bounded linear operators $(T(t))$ on X for

every $t > 0$, the set$Y_\epsilon(t) =\{(\phi_\epsilon x)(t) : x \in Y_0\}$is precompact in X for every ϵ ,$0 < \epsilon < t$
Further , for $x \in Y_0$ we have

$\|(\phi x)(t) - (\phi_\epsilon x)(t)\| \leq \| \int_{t-\epsilon}^{t}(t-s)^{q-1}T_q(t-s)[f(s,u(s) + \int_0^s g(s,\tau,x(\tau),\psi(\tau))d\tau]ds$
$+ \sum_{t-\epsilon<t_k<t} T_q(t-t_k)I_k(x(t_k^-))\|$
$\|(\phi x)(t) - (\phi_\epsilon x)(t)\| \leq \frac{\epsilon^\alpha}{\Gamma(\alpha+1)}M(M_2+M_3) + \sum_{t-\epsilon<t_k<t} L_k(\cdot)$
if $\epsilon > 0$ then there are relatively compact sets close to the set $\{\phi(t) : \phi \in C(J,X)\}$ then$Y_0(t)$ is precompact in X

we will show that $\phi(Y_0) = S = \{(\phi x)(t) : x \in Y_0\}$ is an equicontinuous family of functions for $0 \leq t \leq s$ we have
$\|(\phi x)(t) - (\phi x)(s)\| \leq \|S_q(t) - S_q(s)\|\|x_0\| + \|S_q(t) - S_q(s)\|\|h(x(t_1),...,x(t_p))\|$
$+ \| \int_0^t (t-\tau)^{q-1}T_q(t-\tau) - (s-\tau)^{q-1}T_q(s-\tau)[f(\tau,x(\tau)) + \int_0^\tau g(\tau,\eta,x(\eta),\psi(\eta))d\eta]d\tau]$
$+ \| \int_t^s (s-\tau)^{q-1}T_q(s-\tau)[f(\tau,x(\tau)) + \int_0^\tau g(\tau,\eta,x(\eta),\psi(\eta))d\eta]d\tau]\|$
$+\sum_{0<t_k<s} \|T_q(t-t_k) - T_q(s-t_k)\|\|I_k(x(t_k^-))\| + \sum_{s<t_k<t} \|T_q(t-t_k)\|\|I_k(x(t_k^-))\|$

$$\|(\phi x)(t)-(\phi x)(s)\| \leq \|T_q(t)-T_q(s)\|\|x_0+H\|+(M_2+M_3)\int_0^t \|(t-\tau)^{q-1}T_q(t-\tau)-(s-\tau)^{q-1}T_q(s-\tau)\|d\tau$$

$$+\frac{M}{\Gamma(\alpha+1)}(s-t)^\alpha(M_2+M_3) + \frac{M}{\Gamma(\alpha+1)}\sum_{0<t_k<s}(t-s)\|I_k(x(t_k^-)\| + \frac{M}{\Gamma(\alpha+1)}\sum_{s<t_k<t}\|I_k(x(t_k^-))\|$$

the right hand side of this inequality tends to zero as $s \to t$ then the compactness of $(T(t))_{t\geq 0}$ implies the continuity in the uniform operator topology , then S is bounded in Y by using the (Arzela -Ascoli theorem) S is precompact hence by the Schauder fixed point theorem ϕ has a fixed point in Y_0 and any fixed point is a mild solution of the non-local and impulsive system (1.1)-(1.3)

2.1 Controllability Results

in this section we will introduce a sufficient conditions for controllability of nonlinear fractional integrodifferential system with nonlocal and impulsive conditions in the following form:

$$^cD_0^q[x(t) - F(t,x(t),x(b_1(t)),...,x(b_m(t))] = A[x(t) - F(t,x(t),x(b_1(t)),...,x(b_m(t))]$$
$$+f(t,x(t),x(a_1(t)),...,x(a_n(t))) + Bu(t)$$
$$+\int_0^t g(t,s,x(s),\psi(s))ds,\ t \in J = [0,T], t \neq t_k, k = 1,2,3......,m \quad (4.1)$$

$$x(0) + h(x(t_1),...,x(t_p)) = x_0 \qquad (4.2)$$
$$\Delta x|_{t=t_k} = I_k(x(t_k^-)), k = 1,2,3,......m \qquad (4.3)$$

where the state $x(.)$ takes values in Banach space X and the control function $u(.)$ is given in $L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space .Here B is a bounded linear operator from U into X. For system (4.1), there exist a mild solution of the following form (see [21]):

$x(t) = S_q(t)[x_0-h(x(t_1),...,x(t_p))-F(0,x(0),x(b_1(0)),.....,x(b_m(0)))]+F(t,x(t),x(b_1(t)),...,x(b_m(t))+$

$\int_0^t(t-s)^{q-1}T_q(t-s)[f(s,x(s),x(a_1(s)),......,x(a_n(s))+Bu(s)+\int_0^s g(s,\tau,x(\tau),\psi(\tau))d\tau]ds+\sum_{0<t_k<t} T_q(t-$

$t_k)I_k(x(t_k^-))$is called mild solution of the problem (4.1)-(4.3)

Definition 4.1. System (4.1) is said to be controllable with nonlocal and impulsive conditions on the interval J if , for every $x_0, x_T \in X$ there exist a control function $u \in L^2(J,U)$ such that the mild solution x(.) of (4.1) satisfies
$x(0) + h(x(t_1), ..., x(t_p)) = x_0, x(T) = x_1$
$to establish the result, we need the following additional hypothesis$
$(H6) the linear operator$ W $from$L$^2(J,U)$ into X , defined by
$Wu = \int_0^T (T-s)^{q-1} T_q(T-s)Bu(s)ds \quad u \in L^2(J,U)$ has an inverse operator W^{-1} defined on $L^2(J,U)$ into X , defined on $L^2(J,U)/kerW$ and there exist a constant $M_4 > 0$ such that$\|BW^{-1}\| \leq M_4$

Theorem 4.1 if the hypotheses (H1)-(H6) are satisfied ,then the system (4.1)-(4.4) is controllable on J

Proof. For simplicity we rewrite that

$$(t, x(t), x(b_1(t), ..., x(b_m(t))) = (t, v(t))$$

and

$$(t, x(t), x(a_1(t), ..., x(a_n(t))) = (t, u(t))$$

using the hypothesis (H6) ,for an arbitrary function x(.),define the control

$$u(t) = W^{-1}\{x_T - S_q(t)[x_0 - h(x(t_1), ..., x(t_p)) - F(0, x(0), x(b_1(0)), ..., x(b_m(0)))]+$$
$$F(t, x(t), x(b_1(t)), ..., x(b_m(t)) - \int_0^T (T-s)^{q-1} T_q(t-s)[f(s, x(s), x(a_1(s)), ..., x(a_n(s))+$$
$$\int_0^s g(s, \tau, x(\tau), \psi(\tau))d\tau]ds\}(t)$$

now we will show that , when using this control the operator defined by
$(\phi x)(t) = S_q(t)[x_0 - h(x(t_1), ..., x(t_p)) - F(0, v(0))] + F(t, v(t)) +$
$\int_0^t (t-s)^{q-1} T_q(t-s)[f(s, u(s) + Bu(s) + \int_0^s g(s, \tau, x(\tau), \psi(\tau))d\tau]ds +$
$\sum_{0<t_k<t} T_q(t-t_k)I_k(x(t_k^-))$ this operator has a fixed point. this fixed point is then a solution of (4.1)

since $(\phi x)(T) = x_T$ i.e
$(\phi x)(T) = S_q(t)[x_0 - h(x(t_1), ..., x(t_p)) - F(0, v(0))] + F(t, v(t)) +$
$\int_0^T (T-s)^{q-1} T_q(T-s)BW^{-1} \times \{x_T - S_q(T)[x_0 - h(x(t_1), ..., x(t_p)) - F(0, v(0))] + F(t, v(t)) - \int_0^T (T-\tau)^{q-1} T_q(T-\tau)[f(\tau, u(\tau) + \int_0^\tau g(\tau, \eta, x(\eta), \psi(\eta))d\eta]d\tau\}(\eta)ds + \int_0^t (t-s)^{q-1} T_q(t-s)[f(s, u(s) + \int_0^s g(s, \tau, x(\tau), \psi(\tau))d\tau]ds +$
$\sum_{0<t_k<t} T_q(t-t_k)I_k(x(t_k^-)) = x_T$,which means that the control u steers the semilinear fractional integrodifferential system from the initial state x_0 to final state x_T in time T provided we can obtain a fixed point of the nonlinear operator ϕ
let $Y_0 = \{x \in Y : x(0) + h(x(t_1), ..., x(t_p)) = x_0, \|x(t)\| \leq r', for t \in J\}$
where r' is positive constant . then Y_0 is clearly bounded , closed and convex subset of Y .

Define a mapping $\phi : Y \to Y_0$ by :

$(\phi x)(T) = S_q(t)[x_0 - h(x(t_1), ..., x(t_p)) - F(0, v(0))] + F(t, v(t)) +$
$\int_0^t (t-\eta)^{q-1} T_q(t-\eta)BW^{-1} \times \times \{x_T - S_q(T)[x_0 - h(x(t_1), ..., x(t_p)) - F(0, v(0))] + F(t, v(t)) - \int_0^T (T-s)^{q-1} T_q(T-s)[f(s, u(s) + \int_0^s g(s, \tau, x(\tau), \psi(\tau))d\tau]ds\}d\eta +$

$\int_0^t (t-s)^{q-1} T_q(t-s)[f(s,u(s)+\int_0^s g(s,\tau,x(\tau),\psi(\tau))d\tau]ds +$
$\sum_{0<t_k<t} T_q(t-t_k)I_k(x(t_k^-))$
$\|(\phi x)(T)\| \leq [\|x_0\| + H + M_0L_1(k+1)] + M_0L_1(k+1) + \frac{M_1}{\Gamma(q+1)} t^q(M_2+M_3) + qM_1 \sum_{k=1}^{k=m} \lambda_k +$
$\int_0^t (t-\eta)^{q-1} \|T_q(t-\eta)\| \|BW^{-1}\| \times$
$\{\|x_T\| + \|S_q(t)\|[\|x_0\| + H + M_0L_1(k+1)] + M_0L_1(k+1) +$
$\int_0^T (T-s)^{q-1} \|T_q(T-s)\| [f(s,u(s)+\int_0^s g(s,\tau,x(\tau),\psi(\tau))d\tau]ds\} d\eta +$
$\int_0^t (t-s)^{q-1} \|T_q(t-s)\| [\|f(s,u(s)\| + \int_0^s \|g(s,\tau,x(\tau),\psi(\tau))d\tau\|]ds +$
$\sum_{0<t_k<t} \|T_q(t-t_k)\| \|I_k(x(t_k^-))\|$

$$\|(\phi x)(T)\| \leq M[\|x_0\| + H + M_0L_1(k+1)] + M_0L_1(k+1) + \frac{M}{\Gamma(q+1)} t^q(M_2+M_3) + qM\sum_{k=1}^{k=m} \lambda_k +$$

$$\frac{M}{\Gamma(q+1)} t^q M_4[\|x_T\| + M(\|x_0\| + H + M_0L_1(k+1) + M_0L_1(k+1) + \frac{M}{\Gamma(q+1)} T^q(M_2+M_3)]$$

$$\|(\phi x)(T)\| \leq M\{\|x_0\| + H + M_0L_1(k+1) + M_0L_1(k+1) + \frac{t^q}{\Gamma(q+1)}(M_2+M_3) + q\sum_{k=1}^{k=m} \lambda_k +$$

$$\frac{t^q M_4}{\Gamma(q+1)}[\|x_T\| + M_0L_1(k+1)(1+M) + M(\|x_0\|+H) + \frac{M}{\Gamma(q+1)} T^q(M_2+M_3)]\} + M_0L_1(k+1) = r'$$

since f and g are continuous and $\|(\phi x)(T)\| \leq r'$,it follow that ϕ is continuous and maps Y_0 into itself .moreover ,ϕ maps Y_0 into a precompact subset of Y_0 . To prove this , we first show that every fixed $t \in J$ the set $Y_0(t) = \{(\phi x)(T) : x \in Y_0\}$ is precompact in X. This is clear for $t = 0$ since $Y_0(0)$ is precompact by assumption $(H4)$

let $t > 0$ be fixed and for $0 < \epsilon < t$, define :

$(\phi_\epsilon x)(t) = S_q(t)[x_0 - h(x(t_1),...,x(t_p)) - F(0,v(0))] + F(t,v(t))$
$+ \int_0^{t-\epsilon} (t-\eta)^{q-1} T_q(t-\eta) BW^{-1} \times$
$\{x_T - S_q(T)[x_0 - h(x(t_1),...,x(t_p)) - F(0,v(0))] + F(t,v(t)) -$
$\int_0^T (T-s)^{q-1} T_q(T-s)[f(s,u(s)+\int_0^s g(s,\tau,x(\tau),\psi(\tau))d\tau]ds\}(\eta)d\eta +$
$\int_0^{t-\epsilon} (t-s)^{q-1} T_q(t-s)[f(s,u(s)+\int_0^s g(s,\tau,x(\tau),\psi(\tau))d\tau]ds + \sum_{0<t_k<t} T_q(t-t_k)I_k(x(t_k^-)) + \int_0^{t-\epsilon}(t-$
$s)^{q-1} T_q(t-s)[f(s,u(s)+\int_0^s g(s,\tau,x(\tau),\psi(\tau))d\tau]ds + \sum_{0<t_k<t} T_q(t-t_k)I_k(x(t_k^-))$
since $T(t)$ is compact for $t > 0$, the set $Y_\epsilon(t) = \{(\phi_\epsilon x)(t) : x \in Y_0\}$ is precompact in X for every ϵ , $0 < \epsilon < t$. Furthermore for $x \in Y_0$ we have

$\|(\phi x)(t) - (\phi_\epsilon x)(t)\| \leq \int_{t-\epsilon}^t (t-\eta)^{q-1} \|T_q(t-\eta)\| \|BW^{-1}\| \times$
$\{\|x_T\| + \|s_q(T)\|[\|x_0 + H + A^{-\beta}A^\beta F(0,v(0))] + \|A^{-\beta}A^\beta F(t,v(t))\| +$
$\int_0^T (T-s)^{q-1} \|T_q(T-s)\|[\|f(s,u(s)\| + \|\int_0^s g(s,\tau,x(\tau),\psi(\tau))d\tau\|]ds\}(\eta)d\eta +$
$\int_{t-\epsilon}^t (t-s)^{q-1} \|T_q(t-s)\|[\|f(s,u(s)\| + \|\int_0^s g(s,\tau,x(\tau),\psi(\tau))d\tau\|]ds\} +$
$\sum_{t-\epsilon<t_k<t} \|T_q(t-t_k)\| \|I_k(x(t_k^-))\|$
$\|(\phi x)(t) - (\phi_\epsilon x)(t)\| \leq \frac{MM_4}{\Gamma(q+1)} \epsilon^q \{\|x_T\| + M(\|x_0\| + H + M_0L_1(k+1) +$
$M_0L_1(k+1) + \frac{M\Gamma(q+1)}{T}^q(M_2+M_3)\} + \frac{\epsilon^q}{\Gamma(q+1)}(M_2+M_3) + \sum_{t-\epsilon<t_k<t} T_q(t-t_k)L_k(\cdot)$
If $\epsilon \to 0$ then there exist relatively compact sets close to the set $\phi(t)$ then $y_0(t)$ is precompact in X
We want to show that $\phi(Y_0) = \{\phi x : x \in Y_0\}$ is an equicontinuous family of functions . For that, let $t_2 > t_1 > 0$

$\|(\phi x)(t_1) - (\phi x)(t_2)\| \le \|s_q(t_1) - s_q(t_2)\|[\|x_0\| + H - F(0, v(0))] +$
$\int_0^{t_1} \|T_q(t_1 - \eta)(t_1 - \eta)^{q-1} - T_q(t_2 - \eta)(t_2 - \eta)^{q-1}\|BW^{-1} \times$
$\{x_T - S_q(T)[x_0 - h(x(t_1), ..., x(t_p)) - F(0, v(0))] + F(t, v(t)) -$
$\int_0^T T_q(T - s)(T - s)^{q-1}[f(s, u(s) + \int_0^s g(s, \tau, x(\eta), \psi(\eta))d\eta]d\tau\}(\eta)ds +$
$\int_0^t (t - s)^{q-1}T_q(t - s)[f(s, u(s) + \int_0^s g(s, \tau, x(\tau), \psi(\tau))d\tau]ds\}d\eta +$
$\int_0^{t_1} \|T_q(t_1 - s)(t_1 - s)^{q-1} - T_q(t_2 - s)(t_2 - s)^{q-1}\|[f(s, u(s) +$
$\int_0^s g(s, \tau, x(\eta), \psi(\eta))d\eta]d\tau\}ds - \int_{t_1}^{t_2} \|T_q(t_2 - \eta)(t_2 - \eta)^{q-1}\|BW^{-1} \times$
$\{x_T - S_q(T)[x_0 - h(x(t_1), ..., x(t_p)) - F(0, v(0))] + F(t, v(t)) -$
$\int_0^T T_q(T - s)(T - s)^{q-1}[f(s, u(s) + \int_0^s g(s, \tau, x(\eta), \psi(\eta))d\eta]d\tau\}(\eta)ds\}d\eta -$
$\int_{t_1}^{t_2} \|T_q(t_2 - s)(t_2 - s)^{q-1}\|[f(s, u(s)) + \int_0^s g(s, \tau, x(\eta), \psi(\eta))d\eta]d\tau\}ds +$
$\sum_{0<t_k<t_1} \|T_q(t_2 - t_k) - T_q(t_1 - t_k)\|\|I_k(x(t_k^-))\| +$
$\sum_{t_1<t_k<t_2} \|T_q(t_2 - t_k) - T_q(t_1 - t_k)\|\|I_k(x(t_k^-))\|$
$\|(\phi x)(t_1)-(\phi x)(t_2)\| \le \|s_q(t_1)-s_q(t_2)\|[\|x_0\|+H+M_0L_1(k+1)]+\int_0^{t_1} \|T_q(t_1-\eta)(t_1-\eta)^{q-1}-T_q(t_2-$
$\eta)(t_2-\eta)^{q-1}\| \times M_4\{\|x_T\| + M(\|x_0\| + H + M_0L_1(k+1) + M_0L_1(k+1) + \frac{M}{\Gamma(q+1)}T^q(M_2+M_3)\}d\eta+$
$\int_0^{t_1} \|T_q(t_1 - s)(t_1 - s)^{q-1} - T_q(t_2 - s)(t_2 - s)^{q-1}\|(M_2 + M_3)ds + \frac{MM_4(t_2-t_1)^q}{\Gamma(q+1)}[\|x_T\| + M(\|x_0\| +$
$H + M_0L_1(k + 1)) + M_0L_1(k + 1) +$
$\frac{M_1(T)^q}{\Gamma(q+1)}(M_2 + M_3) + \sum_{0<t_k<t_1} \|T_q(t_2 - t_k) - T_q(t_1 - t_k)\|\|I_k(x(t_k^-))\| +$
$\sum_{t_1<t_k<t_2} \|T_q(t_2 - t_k) - T_q(t_1 - t_k)\|\|I_k(x(t_k^-))\|$

the compactness of $S_\alpha(t), T_\alpha(t), t > 0$ and $(T(t))_{t\ge 0}$ is continuous in the uniform operator topology.

The right hand side is independent of $x \in Y_0$ tends to zero as $t_2 \to t_1$ then $\phi(Y_0)$ is equicontinuous family of functions

$\phi(Y_0)$ is bounded in Y and so by (Arzela-Ascoli) theorem , $\phi(y_0$ is precompact hence from the Schauder fixed point theorem ϕ has a fixed point in Y_0

any fixed point of ϕ is a mild solution of the system we are study it on J satisfying $(\phi x)(t) = x(t) \in X$ then the system (4.1)is controllable on J

5. Example

Consider the fractional nonlocal impulsive integro-partial differential control system of the form
$\frac{\partial^q}{\partial t^q}[w(x, t) - F(t, w(x, t), w(x(b_1(t), ..., x(b_m(t), t)] =$
$a(x, t, w(x, t))\frac{\partial^2}{\partial x^2}[w(x, t)-F(t, w(x, t), w(x(b_1(t), ..., x(b_m(t), t)]+\zeta(x, t)+\Psi(t, x \arctan \phi(x, t, w))+$

$\int_0^t e^{-\phi(x,s,w)}ds \qquad (5.1)$
$w(x, 0) + \sum_{k=1}^m c_k w(x, t_k) = w_0(x), x \in [0, \pi] \qquad (5.2)$
$w(0, t) = w(\pi, t) = 0, t \in J \qquad (6.3)$
$\Delta w(t_k, x) = -w(t_k, x), x \in (0, 1), k = 1, ..., m \qquad (5.4)$
where $\alpha \le 1, 0 < t_1 < ..... < t_m < T$ the functions Ψ ϕ and $a(x, t, w(x, t))$ are continuous functions.
let us take

$X = L^2[0, \pi]$, $PC = PC(J, s_\delta)$, $s_\delta = \{y \in L^2[0, \pi] : \|y\| \le \delta\}$
put $(Bw)(x, t) = \zeta(x, t), x \in (0, \Pi)$ where $w(t) = \zeta(., t)$ and $\zeta : [0, \pi] \times J \to [0, \pi]$ is continuous
$h(w(., t)) = \sum_{k=1}^m c_k w(., t_k)$ the functions $f(t, w(x, t), w(x(b_1(t), ..., x(b_m(t), t) = x \arctan \phi(x, t, w))$

and $g(t, s, x(s), \Psi(s))ds = e^{-\phi(x,s,w)}$

we define $A(t, .) : X \to X$ by $(A(t, .)w)(x) = a(x, t, .)\acute{w}$ with$w \in D(A)$
since the operator A is the infinitesimal generator of an compact analytic semigroup $T(t)$
the equation (6.1) can be reformulated as the following abstract equation in X :

${}^{c}D_0^q[u(t) - F(t, u(t), u(b_1(t)), ..., u(b_m(t))] = A[u(t) - F(t, u(t), u(b_1(t)), ..., u(b_m(t))] +$
$f(t, u(t), u(a_1(t)), ..., u(a_n(t)))+Bu(t)+\int_0^t g(t, s, x(s), \psi(s))ds,\ t \in J = [0, T], t \neq t_k, k = 1, 2, 3......, m$
that is $u(t) - F(t, u(t), u(b_1(t)), ..., u(b_m(t)) = w(t, .)$ and $u(t)(x) = w(t, x), t \in [0, T]$
$f : [0, T] \times X \to X$ is given by

$f(t, u(t), u(a_1(t)), ..., u(a_n(t))) = f(t, x \arctan w(u, t)), g(u) = e^{-\phi(x,s,w)}$ since the functions $f(t, u)$ and $g(u)$ are continuous from the previous conditions (H1)-(H5) on X
assume that the linear operator W is given by :

$Wu(x) = \int_0^T (T-s)^{q-1}T_q(T-s)Bu(s)ds$ has a bounded invertible operator W^{-1} in $L^2(J, U)/kerW$ further all conditions (H1)-(H5) are satisfied hence by using theorem (4.1) the system (5.1)-(5.3) is controllable on J.

3 Conclusions

In this paper, we have presented by using semigroup and Schauder fixed point theorem the existence of mild solutions of fractional integrodifferential equations with nonlocal and impulsive conditions in Banach spaces also sufficient conditions for controllability of fractional integrodifferential systems are established.

Acknowledgment

I would like to thank the referees and Editor for their valuable comments and suggestions.

Competing Interests

The author declares that no competing interests exist.

References

[1] Hilfer R. Applications of fractional calculus in physics. World Scientific, Singapore; 2000.

[2] Pengyu Chen. On the initial value problem of fractional evolution equations with noncompact semigroup. Comput. Math. Appl. 2014;67:1108-1115.

[3] Pengyu Chen. Existence of mild solutions for fractional evolution equations with mixed monotone nonlocal conditions. Zeitschrift fur Angewandte Mathematik und Physik. 2014;65:711-728.

[4] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. Elsevier, Amsterdam; 2006.

[5] Lakshmikantham V, Leela S, Vasundhara Devi J. Theory of fractional dynamic systems. Cambridge Academic Publishers, Cambridge; 2009.

[6] Metzler F, Schick W, Kilian HG, Nonnenmacher TF. Relaxation in filled polymers: A fractional calculus approach. Journal of Chemical Physics. 1995;103:71807186.

[7] Deepmala, Pathak HK. Study on existence of solutions for some nonlinear functional-integral equations with applications. Math. Commun. 2013;18: 97-107.

[8] Miller KS, Ross B. An introduction to the fractional calculus and fractional differential equations. Wiley, New York; 1993.

[9] Podlubny I. Fractional differential equations. Academic Press, New York; 1999.

[10] Radhakrishnan B, Balachandran K. Controllability of impulsive Neutral functional evolution integrodifferential systems with infinite delay. Nonlinear Analysis: Hybrid Systems. 2011;5:655-670.

[11] Hamdy M. Ahmed. Controllability for sobolev type fractional integro-differential systems in a banach space. Ahmed Advances in Difference Equations. 2012; 2012:167.

[12] Radhakrishnan B, Balachandran K. Controllability of nonlocal impulsive functional integrodifferential evolution systems. The Journal of Nonlinear Science and Applications. 2011;4:281-291.

[13] Radhakrishnan B, Balachandran K. Controllability results for nonlinear impulsive integrodifferential evolution systems with time varying delays. Journal of Control Theory and Applications. 2013;11(4):415-421.

[14] Radhakrishnan B, Balachandran K. Controllability results for semilinear impulsive integrodifferential evolution systems with nonlocal conditions. Journal of Control Theory and Applications. 2012;10:28-34.

[15] Hamdy M. Ahmed, Hassan AAM, Ghanem AS. Existence of mild solution for impulsive fractional differential equations with non-local conditions in banach space. British Journal of Mathematics and Computer Science. 2014;4(6):830-840.

[16] blachandran K, kiruthika S. Existence results for fractional integro differntial equations with nonlocal condition via resolvent operators. Computers and Mathematics with Applications. 2011;1350-1358.

[17] Yong Zhou, Feng Jiao. Existence of mild solutions for fractional neutral evolution equations, computer and mathematics with applications. 2010;59:1063-1077.

[18] Pengyu Chen, Yongxiang Li, Qiyu Chen, Binhua Feng. On the initial value problem of fractional evolution equations with noncompact semigroup. Computers and Mathematics with Applications. 2014;67:1108-1115.

[19] Pengyu Chen, Yongxiang Li. Existence of mild solutions for fractional evolution equations with nonlocal initial conditions. Zeitschrift fr Angewandte Mathematik und Physik. 2014;65:711-728.

[20] Pengyu Chen. Existence of mild solutions for fractional evolution equations with non-local initial conditions. Ann. Polon. Math. 2014;110:13-24.

[21] Hamdy M. Ahmed. Non-linear fractional integro-differential systems with non-local conditions. IMA Journal of Mathematical Control and Information; 2014.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/10313