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Traveling Wave Solutions of the Fourth Order Boussinesq Equation via the Improved (G'/G) Expansion Method

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Authors' contributions

The work presented here was carried out in collaboration between three authors. Author ASA carried out the main part of this manuscript. Author MMZ discussed and polished the manuscript. Author SS data analyzed and polished the manuscript. All authors have read and approved the manuscript.

Research Article

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ABSTRACT

In this article, the fourth order Boussinesq equation is investigated to generate many new traveling wave solutions via the improved (G'/G)-expansion method. In the method the second order linear ordinary differential equation with constant coefficients is used. Also, the resulted solutions are presented in three different families including solitons and periodic solutions. Furthermore, some of our solutions are coincided with published results which gained by other authors and some are new.
Aims: The aim of this work is to construct many new exact traveling wave solutions including solitons, periodic and rational solutions of the fourth order Boussinesq equation by applying the improved (G'/G)-expansion method.
Methodology: The improved (G'/G)-expansion method is effective and powerful mathematical tool for solving nonlinear partial differential equations which arise in mathematical physics, engineering sciences and other technical arena. In addition,

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$G'' + \lambda G' + \mu G = 0,$ together with $f(\varphi) = \sum_{h=-p}^{p} a_h (G'/G)^h$, is implemented as traveling wave solutions, where either a_{-p} or a_p may be zero, but both a_{-p} and a_p cannot be zero at the same time.

Results: The obtained traveling wave solutions are described in terms of the hyperbolic functions, the trigonometric functions and the rational functions.

Conclusion: The constructed solutions may express a variety of new features of waves, further, may be valuable in the theoretical and numerical studies of the considered equation. Moreover, the obtained exact solutions reveal that the improved (G'/G)-expansion method is a promising mathematical tool, because, it can establish abundant new traveling wave solutions of different physical structures. Also, some of our solutions are in good agreement with already published results for a special case and others are new.

Keywords: The improved-expansion method; the Boussinesq equation; traveling wave solutions; nonlinear evolution equations.

1. INTRODUCTION

The study of nonlinear evolution equations (NLEEs) has now become one of the most exciting and tremendously active research areas to scientists. In recent time, many authors who are interested in the nonlinear complex physical phenomena generated analytical solutions of NLEEs. They introduced various methods, for instance, the inverse scattering method [1], the homogeneous balance method [2], the Hirota's bilinear transformation method [3,4], the Jacobi elliptic function expansion method [5-7], the generalized Riccati equation method [8], the tanh-coth method [9-11], the F-expansion method [12], the Exp-function method [13-15] and others [16-18].

Wang et al. [19] presented a powerful method which is called the (G'/G)-expansion method. They generated exact traveling wave solutions in three different families including solitons, periodic and rational solutions for some nonlinear partial differential equations (PDEs) via this method. In the method, $u(\xi) = \sum_{i=0}^{m} a_i (G'/G)^i$ is implemented as traveling wave solutions, where $a_m \neq 0$. Later on, several scientists investigated various nonlinear PDEs to construct traveling wave solutions via this method [20-26]. More recently, Zhang et al. [27] extended this method and called the improved (G'/G)-expansion method. They used $f(\varphi) = \sum_{h=-p}^{p} a_h (G'/G)^h$ as traveling wave solutions, where either a_{-p} or a_p may be zero, but both a_{-p} and a_p cannot be zero simultaneously. After that, many researchers studied different NLEEs to construct exact solutions by using this improved (G'/G)-expansion method. For example, Zhao *et al.* [28] executed the same method to establish exact solutions of the variant Boussinesq equations. Nofel et al. [29] constructed traveling wave solutions for the fifth-order KdV equation by using this method. Hamad et al. [30] studied higher dimensional potential YTSF equation to obtain analytical solutions via the same method. Naher and Abdullah [31] applied this powerful method to construct traveling wave solutions of the nonlinear reaction diffusion equation whilst they [32] implemented this method to generate exact solutions of the (2+1)-dimensional Modified Zakharov-Kuznetsov equation via the improved (G'/G)**-**expansion method. In Ref. [33] Naher *et al*. generated many solutions of the compound KdV-Burgers equation via the improved (G'/G)-expansion method and so on.

In this paper, we would like to investigate well established fourth order Boussinesq equation by applying the improved (G'/G)-expansion method to construct a rich class of new exact traveling wave solutions including solitons, periodic and rational solutions.

2. DESCRIPTION OF THE IMPROVED (G'/G)-EXPANSION METHOD

Let us consider the general nonlinear partial differential equation:

$$P\left(u,u_t,u_x,u_{xt},u_{tt},u_{xx},...\right)=0, \tag{1}$$

where $u=u(x,t)$ is an unknown function, P is a polynomial in $u(x,t)$ and its partial derivatives in which the highest order partial derivatives and the nonlinear terms are involved.

The main steps of the method are as follows:

Step 1. Suppose the traveling wave variable:

$$u(x,t)=f(\varphi), \quad \varphi=x-Vt, \tag{2}$$

where V is the speed of the traveling wave. Using Eq. (2), Eq. (1) is transformed into an ordinary differential equation for $f=f(\varphi)$:

$$Q(f,f',f'',f''',...)=0, \tag{3}$$

where Q is a function of $f(\varphi)$ and its total derivatives.

Step 2. Eq. (3) can be integrated term by term one or more times, according to possibility, yields constant(s) of integration. The integral constant may be zero, for simplicity.

Step 3. Suppose that the traveling wave solution of Eq. (3) can be stated in the following form:

$$f(\varphi)=\sum_{h=-p}^{p} a_h\left(G'/G\right)^h \tag{4}$$

with $G=G(\varphi)$ satisfies the second order linear ODE:

$$G'' + \lambda G' + \mu G = 0, \tag{5}$$

where $a_h \left(h = 0, \pm 1, \pm 2, ..., \pm\ p\right)$, λ and μ are constants.

Step 4. The value of the positive integer p can be determined by taking the homogeneous balance between the highest order nonlinear terms and the highest order derivatives appearing in Eq. (3).

Step 5. Substitute Eq. (4) together with Eq. (5) into Eq. (3) with the value of p obtained in Step 4. Equating the coefficients of $\left(G'/G\right)^r$, $\left(r = 0, \pm 1, \pm 2, ...\right)$, then setting each coefficient to zero, we obtain a set of algebraic equations for $a_h \left(h = 0, \pm 1, \pm 2, ..., \pm\ p\right), V, \lambda$ and μ.

Step 6. Solve the system of algebraic equations which are obtained in step 5 with the aid of commercial software Maple and we obtain values for $a_h \left(h = 0, \pm 1, \pm 2, ..., \pm\ p\right), V, \lambda$ and μ. Then, substitute obtained values in Eq. (4) along with the general solution of Eq. (5) with the value of p, we can obtain the traveling wave solutions of Eq. (1).

2.1 Application of the Method

In this subsection, we have studied the fourth order Boussinesq equation to construct exact traveling wave solutions via the improved $\left(G'/G\right)$ **-**expansion method.

2.1.1 The fourth order Boussinesq equation

We consider the fourth order Boussinesq equation followed by Naher and Abdullah [36]:

$$u_{tt} - \alpha^2 u_{xx} - \beta\left(u^2\right)_{xx} + u_{xxxx} = 0. \tag{6}$$

Now, we use the wave transformation Eq. (2) into the Eq. (6), therefore, integrating with respect to φ twice and setting the constants of integration to zero, yields:

$$\left(V^2 - \alpha^2\right) f - \beta f^2 + f'' = 0. \tag{7}$$

Taking the homogeneous balance between f^2 and f'' in Eq. (7), we obtain $p = 2$. Therefore, the solution of Eq. (7) is of the form:

$$f\left(\varphi\right) = a_{-2}\left(G'/G\right)^{-2} + a_{-1}\left(G'/G\right)^{-1} + a_0 + a_1\left(G'/G\right) + a_2\left(G'/G\right)^2, \tag{8}$$

where a_{-2}, a_{-1}, a_0, a_1 and a_2 are constants to be determined.

Substituting Eq. (8) together with Eq. (5) into the Eq. (7), the left-hand side of Eq. (7) is converted into a polynomial of $\left(G'/G\right)^r, \left(r = 0, \pm 1, \pm 2, ...\right)$. According to Step 5, collecting all terms with the same power of $\left(G'/G\right)$. Then, setting each coefficient of the resulted polynomial to zero, yields a set of algebraic equations (for simplicity, which are not presented) for $a_{-2}, a_{-1}, a_0, a_1, a_2, V, \lambda$ and μ. Solving the system of obtained algebraic equations with the help of algebraic software Maple 13, we obtain four different values.

Case 1:

$$a_{-2} = 0, a_{-1} = 0,\ a_0 = \frac{6\mu}{\beta},\ a_1 = \frac{6\lambda}{\beta}, a_2 = \frac{6}{\beta},\ V = \pm\sqrt{\alpha^2 - \left(\lambda^2 - 4\mu\right)}, \tag{9}$$

where α, β, λ and μ are free parameters.

Case 2:

$$a_{-2} = 0, a_{-1} = 0,\ a_0 = \frac{2\mu + \lambda^2}{\beta},\ a_1 = \frac{6\lambda}{\beta}, a_2 = \frac{6}{\beta},\ V = \pm\sqrt{\alpha^2 + \left(\lambda^2 - 4\mu\right)}, \tag{10}$$

where α, β, λ and μ are free parameters.

Case 3:

$$a_{-2} = \frac{6\mu^2}{\beta}, a_{-1} = \frac{6\lambda\mu}{\beta},\ a_0 = \frac{6\mu}{\beta},\ a_1 = 0, a_2 = 0,\ V = \pm\sqrt{\alpha^2 - \left(\lambda^2 - 4\mu\right)}, \tag{11}$$

where α, β, λ and μ are free parameters.

Case 4:

$$a_{-2} = \frac{6\mu^2}{\beta}, a_{-1} = \frac{6\lambda\mu}{\beta},\ a_0 = \frac{2\mu + \lambda^2}{\beta},\ a_1 = 0, a_2 = 0,\ V = \pm\sqrt{\alpha^2 + \left(\lambda^2 - 4\mu\right)}, \tag{12}$$

where α, β, λ and μ are free parameters.

Substituting the general solution Eq. (5) into Eq. (8), we obtain three different families of traveling wave solutions of Eq. (7):

Family 1: Hyperbolic function solutions:

When $\lambda^2 - 4\mu > 0$, we obtain

$$\begin{aligned} f(\varphi) = {} & a_{-2}\left(\frac{-\lambda}{2}+\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\frac{P\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\varphi+Q\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\varphi}{P\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\varphi+Q\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\varphi}\right)^{-2} \\ & + a_{-1}\left(\frac{-\lambda}{2}+\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\frac{P\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\varphi+Q\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\varphi}{P\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\varphi+Q\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\varphi}\right)^{-1} + a_0 \\ & + a_{1}\left(\frac{-\lambda}{2}+\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\frac{P\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\varphi+Q\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\varphi}{P\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\varphi+Q\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\varphi}\right) \\ & + a_{2}\left(\frac{-\lambda}{2}+\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\frac{P\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\varphi+Q\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\varphi}{P\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\varphi+Q\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\varphi}\right)^{2}, \end{aligned} \tag{13}$$

if P and Q are taken particular values, various known solutions can be rediscovered.

Family 2: Trigonometric function solutions:

When $\lambda^2-4\mu<0,$ we obtain

$$\begin{aligned} f(\varphi) = {} & a_{-2}\left(\frac{-\lambda}{2}+\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\frac{-P\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\varphi+Q\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\varphi}{P\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\varphi+Q\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\varphi}\right)^{-2} \\ & + a_{-1}\left(\frac{-\lambda}{2}+\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\frac{-P\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\varphi+Q\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\varphi}{P\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\varphi+Q\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\varphi}\right)^{-1} + a_0 \\ & + a_{1}\left(\frac{-\lambda}{2}+\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\frac{-P\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\varphi+Q\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\varphi}{P\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\varphi+Q\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\varphi}\right) \\ & + a_{2}\left(\frac{-\lambda}{2}+\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\frac{-P\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\varphi+Q\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\varphi}{P\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\varphi+Q\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\varphi}\right)^{2}, \end{aligned} \tag{14}$$

if P and Q are taken particular values, various known solutions can be rediscovered.

Family 3: Rational solution:

When $\lambda^2-4\mu=0,$ we obtain

$$f(\varphi) = a_{-2}\left(\frac{-\lambda}{2} + \frac{Q}{P+Q\varphi}\right)^{-2} + a_{-1}\left(\frac{-\lambda}{2} + \frac{Q}{P+Q\varphi}\right)^{-1} + a_0 + a_1\left(\frac{-\lambda}{2} + \frac{Q}{P+Q\varphi}\right) + a_2\left(\frac{-\lambda}{2} + \frac{Q}{P+Q\varphi}\right)^{2}. \tag{15}$$

Substituting Eqs. (9), (10), (11) and (12) together with the general solution Eq. (5) into the Eq. (8), we obtain the hyperbolic function solution Eq. (13), then our traveling wave solutions become respectively (if $P=0$ but $Q \neq 0$):

$$f_1(\varphi) = \frac{3(\lambda^2-4\mu)}{2\beta}\left(\left(\coth\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \varphi\right)^2 - 1\right),$$

where $\varphi = x \mp \sqrt{\alpha^2 - (\lambda^2-4\mu)}\,t$.

$$f_2(\varphi) = \frac{(\lambda^2-4\mu)}{2\beta}\left(3\left(\coth\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \varphi\right)^2 - 1\right),$$

where $\varphi = x \mp \sqrt{\alpha^2 + (\lambda^2-4\mu)}\ t$.

$$f_3(\varphi) = \frac{6\mu}{\beta}\left(\mu\left(\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\coth\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \varphi\right)^{-2} + \lambda\left(\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\coth\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \varphi\right)^{-1} + 1\right),$$

where $\varphi = x \mp \sqrt{\alpha^2 - (\lambda^2-4\mu)}\ t$.

$$f_4(\varphi) = \frac{1}{\beta}\left(6\mu\left(\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\coth\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \varphi\right)^{-2} + \lambda\mu\left(\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\coth\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \varphi\right)^{-1} + 2\mu + \lambda^2\right),$$

where $\varphi = x \mp \sqrt{\alpha^2 + (\lambda^2-4\mu)}\ t$.

Again, substituting Eqs. (9), (10), (11) and (12) together with the general solution Eq. (5) into the Eq. (8), we obtain the hyperbolic function solution Eq. (13), then we obtain following solutions respectively (if $Q=0$ but $P \neq 0$):

$$f_5(\varphi) = \frac{3(\lambda^2-4\mu)}{2\beta}\left(\left(\tanh\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \varphi\right)^2 - 1\right),$$

$$f_6(\varphi) = \frac{(\lambda^2-4\mu)}{2\beta}\left(3\left(\tanh\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \varphi\right)^2 - 1\right),$$

$$f_7(\varphi) = \frac{6\mu}{\beta}\left(\mu\left(\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\tanh\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \varphi\right)^{-2} + \lambda\left(\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\tanh\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \varphi\right)^{-1} + 1\right),$$

$$f_8(\varphi) = \frac{1}{\beta}\left(6\mu\left(\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\tanh\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \varphi\right)^{-2} + \lambda\mu\left(\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\tanh\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \varphi\right)^{-1} + 2\mu + \lambda^2\right),$$

Again, substituting Eqs. (9), (10), (11) and (12) together with the general solution Eq. (5) into the Eq. (8), yields the trigonometric function solution Eq. (14), then our obtain solutions become respectively (if $P=0$ but $Q\neq 0$):

$$f_9(\varphi)=\frac{3\left(4\mu-\lambda^2\right)}{2\beta}\left(\left(\cot\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \varphi\right)^2+1\right),$$

where $\varphi=x\mp\sqrt{\alpha^2-\left(\lambda^2-4\mu\right)}\,t.$

$$f_{10}(\varphi)=\frac{\left(4\mu-\lambda^2\right)}{2\beta}\left(3\left(\cot\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \varphi\right)^2+1\right),$$

where $\varphi=x\mp\sqrt{\alpha^2+\left(\lambda^2-4\mu\right)}\,t.$

$$f_{11}(\varphi)=\frac{6\mu}{\beta}\left(\mu\left(\frac{-\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\cot\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \varphi\right)^{-2}+\lambda\left(\frac{-\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\cot\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \varphi\right)^{-1}+1\right),$$

where $\varphi=x\mp\sqrt{\alpha^2-\left(\lambda^2-4\mu\right)}\ t.$

$$f_{12}(\varphi)=\frac{1}{\beta}\left(6\mu\left(\frac{-\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\cot\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \varphi\right)^{-2}+\lambda\mu\left(\frac{-\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\cot\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \varphi\right)^{-1}+2\mu+\lambda^2\right),$$

where $\varphi=x\mp\sqrt{\alpha^2+\left(\lambda^2-4\mu\right)}\ t.$

Also, substituting Eqs. (9), (10), (11) and (12) together with the general solution Eq. (5) into the Eq. (8), yields the trigonometric function solution Eq. (14), our traveling wave solutions become respectively (if $Q=0$ but $P\neq 0$):

$$f_{13}(\varphi)=\frac{3\left(4\mu-\lambda^2\right)}{2\beta}\left(\left(\tan\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \varphi\right)^2+1\right).$$

$$f_{14}(\varphi)=\frac{\left(4\mu-\lambda^2\right)}{2\beta}\left(3\left(\tan\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \varphi\right)^2+1\right).$$

$$f_{15}(\varphi)=\frac{6\mu}{\beta}\left(\mu\left(\frac{-\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\tan\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \varphi\right)^{-2}+\lambda\left(\frac{-\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\tan\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \varphi\right)^{-1}+1\right).$$

$$f_{16}(\varphi)=\frac{1}{\beta}\left(6\mu\left(\frac{-\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\tan\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \varphi\right)^{-2}+\lambda\mu\left(\frac{-\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\tan\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \varphi\right)^{-1}+2\mu+\lambda^2\right).$$

Family 3: Rational solutions:

Substituting Eqs. (9), (10), (11) and (12) together with the general solution Eq. (5) into the Eq. (8), we obtain the rational function solution Eq. (15), and our solutions become respectively (if $\lambda^2-4\mu=0$):

$$f_{17}(\varphi)=\frac{3}{2\beta}\left(\left(\frac{2Q}{(P+Q\varphi)}\right)^{2}-\left(\lambda^{2}-4\mu\right)\right),$$

where $\varphi = x \mp \sqrt{\alpha^2-\left(\lambda^2-4\mu\right)}\,t.$

$$f_{18}(\varphi)=\frac{1}{2\beta}\left(3\left(\frac{2Q}{(P+Q\varphi)}\right)^{2}-\left(\lambda^{2}-4\mu\right)\right),$$

where $\varphi = x \mp \sqrt{\alpha^2+\left(\lambda^2-4\mu\right)}\,t.$

$$f_{19}(\varphi)=\frac{6\mu}{\beta}\left(\mu\left(\frac{-\lambda}{2}+\frac{Q}{P+Q\varphi}\right)^{-2}+\lambda\left(\frac{-\lambda}{2}+\frac{Q}{P+Q\varphi}\right)^{-1}+1\right),$$

where $\varphi = x \mp \sqrt{\alpha^2-\left(\lambda^2-4\mu\right)}\,t.$

$$f_{20}(\varphi)=\frac{6\mu}{\beta}\left(\mu\left(\frac{-\lambda}{2}+\frac{Q}{P+Q\varphi}\right)^{-2}+\lambda\left(\frac{-\lambda}{2}+\frac{Q}{P+Q\varphi}\right)^{-1}\right)+\frac{2\mu+\lambda^{2}}{\beta},$$

where $\varphi = x \mp \sqrt{\alpha^2+\left(\lambda^2-4\mu\right)}\,t.$

3. RESULTS AND DISCUSSION

Many researchers solved the fourth-order Boussinesq equation for obtaining analytical solutions by using different methods. For instance, Zhang [34] studied this equation by applying the F-expansion method to obtain exact solutions. Wazwaz [35] executed the extended tanh method to establish analytical solutions of the same equation while Naher and Abdullah [36] investigated this equation via the basic (G'/G)**-**expansion method to construct traveling wave solutions. To the best of our knowledge, this equation is not solved via the improved (G'/G)**-**expansion method. We have studied mentioned equation by applying the improved (G'/G)**-**expansion method and abundant solutions are constructed in this article. Moreover, it is worth declaring that some of our obtained solutions are in good agreement with already published results and others have not been reported in the literature. Our solutions $f_1(\varphi), f_2(\varphi), f_5(\varphi)$, $f_6(\varphi), f_9(\varphi), f_{10}(\varphi), f_{13}(\varphi), f_{14}(\varphi)$, $f_{17}(\varphi)$ and $f_{18}(\varphi)$ are coincided with the published results $A_1(\psi), A_2(\psi), A_3(\psi)$, $A_4(\psi)$, $A_7(\psi), A_8(\psi), A_9(\psi), A_{10}(\psi), A_{13}(\psi)$ and $A_{14}(\psi)$ respectively, which are gained by Naher and Abdullah [36]. Beyond this, we obtain new exact traveling wave solutions $f_3(\varphi), f_4(\varphi), f_7(\varphi), f_8(\varphi), f_{11}(\varphi), f_{12}(\varphi), f_{15}(\varphi), f_{16}(\varphi), f_{19}(\varphi)$ and $f_{20}(\varphi)$ which are not being established in the previous literature.

4. CONCLUSION

In this article, an improved (G'/G)**-**expansion method is applied to generate new traveling wave solutions of nonlinear partial differential equation, namely, the Boussinesq equation. The obtained solutions are presented through the hyperbolic functions, the trigonometric functions and the rational functions. Moreover, our constructed solutions show that the solution procedure of this method is very simple, reliable and straightforward. In addition, this method can give many new solutions at a time than the basic (G'/G)-expansion method. We hope, this powerful and effective, improve (G'/G)-expansion method could lead to find abundant new traveling wave solutions for various nonlinear partial differential equations which frequently arise in real time technical arena.

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COMPETING INTERESTS

The authors declare that they have no competing interests.

REFERENCES

1. Ablowitz MJ, Clarkson PA. Solitons, nonlinear evolution equations and inverse scattering transform. Cambridge Univ. Press, Cambridge; 1991.
2. Wang ML, Zhou YB, Li ZB. Application of homogeneous balance method to exact solutions of nonlinear equations in mathematical physics. Pyss. Lett. A. 1996;216:67-75.
3. Hirota R. Exact solution of the KdV equation for multiple collisions of solutions. Physics Review Letters. 1971;27:1192-1194.
4. Yan J. Soliton resonances of the nonisospectral modified Kadomtsev-Petviashvili equation. Applied Mathematics. 2011;2:685-693.
5. Liu S, Fu Z, Liu S, Zhao Q. Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations. Physics Letters A. 2001;289:69–74.
6. Chen Y, Wang Q. Extended Jacobi elliptic function rational expansion method and abundant families of Jacobi elliptic function solutions to (1+1)-dimensional dispersive long wave equation. Chaos, Solit. Fract. 2005;24(3):745-757.
7. Xiang C. Jacobi elliptic function solutions for (2+1) dimensional Boussinesq and Kadomtsev-Petviashvili equation, Applied Mathematics. 2011;2:1313-1316.
8. Yan Z, Zhang HQ. New explicit solitary wave solutions and periodic wave solutions for Whitham-Broer-Kaup equation in shallow water. Phy. Lett. A. 2001;285:355-362.
9. Malfliet W. Solitary wave solutions of nonlinear wave equations. Am. J. Phys. 1992;60:650–654.
10. Wazwaz AM. The tanh-coth method for solitons and kink solutions for nonlinear parabolic equations. Appl. Math. and Comput. 2007;188:1467-1475.
11. Abdel-All NH, Abdel Razek MA, Seddeek AK. Expanding the tanh-function method for solving nonlinear equations. Applied Mathematics. 2011;2:1096-1104.
12. Wang ML, Li XZ. Applications of F-expansion to periodic wave solutions for a new Hamiltonian amplitude equation. Chaos, Solitons and Fractals. 2005;24:1257-1268.
13. He JH, Wu XH. Exp-function method for nonlinear wave equations. Chaos Solitons and Fractals. 2006;30:700-708.

14. Naher H, Abdullah FA, Akbar MA. New traveling wave solutions of the higher dimensional nonlinear partial differential equation by the Exp-function method. Journal of Applied Mathematics. Article ID: 575387, 14 pages, doi: 10.1155/2012/575387. 2012.
15. Kumar A. Analytical and numerical description of some nonlinear evolution equations. Physical Review & Research International. 2012;2(4):144-152.
16. Liu Q. Xu R. Periodic solutions of a cohen-grossberg-type BAM neural networks with distributed delays and impulses. Journal of Applied Mathematics, Article ID 643418, 17 pages, doi: 10.1155/2012/643418, 2012.
17. Noor M, Noor K, Waheed A, Al-Said EA. An efficient method for solving system of third-order nonlinear boundary value problems. Mathematical Problems in Engineering, Article ID 250184, 14 pages, doi: 10.1155/2011/250184, 2011.
18. Alexeev BV, Ovchinnikova IV. Application of non-local quantum hydrodynamics to the description of the charge density waves in the graphene crystal lattice. Physical Review and Research International. 2013;3(2):55-116.
19. Wang M, Li X, Zhang J. The $\left(G'/G\right)$-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. Physics Letters A. 2008;372: 417-423.
20. Feng J, Li W, Wan Q. Using $\left(G'/G\right)$-expansion method to seek traveling wave solution of Kolmogorov-Petrovskii-Piskunov equation. Applied Mathematics and Computation, 2011;217:5860-5865.
21. Naher H, Abdullah FA, Akbar MA. The $\left(G'/G\right)$-expansion method for abundant traveling wave solutions of Caudrey-Dodd-Gibbon equation. Mathematical Problems in Engineering, Article ID: 218216(2011), 11 pages, doi:10.1155/2011/218216.
22. Abazari R, Abazari R. Hyperbolic, trigonometric and rational function solutions of Hirota-Ramani equation via $\left(G'/G\right)$-expansion method. Mathematical Problems in Engineering, Article ID: 424801, 11 pages, doi:10.1155/2011/424801, 2011.
23. Zayed EME, Al-Joudi S. Applications of an Extended $\left(G'/G\right)$-Expansion Method to Find Exact Solutions of Nonlinear PDEs in Mathematical Physics. Mathematical Problems in Engineering, Article ID 768573(2010), 19 pages, doi:10.1155/2010/768573.
24. Jabbari A, Kheiri H, Bekir A. Exact solutions of the coupled Higgs equation and the Miccari system using He's semi-inverse method and $\left(G'/G\right)$-expansion method. Computers and Mathematics with Applications. 2011;62:2177-2186.
25. Ozis T, Aslan I. Application of the $\left(G'/G\right)$-expansion method to Kawahara type equations using symbolic computation. Applied Mathematics and Computation. 2010;216:2360-2365.
26. Borhanifar A, Z.Moghanlu A. Application of the $\left(G'/G\right)$-expansion method for the Zhiber-Sabat equation and other related equations. Mathematical and Computer Modelling. 2011;54:2109-2116.
27. Zhang J, Jiang F, Zhao X. An improved $\left(G'/G\right)$-expansion method for solving nonlinear evolution equations. International Journal of Computer Mathematics. 2010;87(8):1716-1725.

28. Zhao YM, Yang YJ, Li W. Application of the improved (G'/G) -expansion method for the Variant Boussinesq equations. Applied Mathematics Sciences. 2011;5(58)2855-2861.
29. Nofel TA, Sayed M, Hamad YS, Elagan SK. The improved (G'/G) -expansion method for solving the fifth-order KdV equation. Annals of Fuzzy Mathematics and Informatics. 2011;3(1):9-17.
30. Hamad YS, Sayed M, Elagan SK, El-Zahar ER. The improved (G'/G) -expansion method for solving (3+1)-dimensional potential-YTSF equation. Journal of Modern Methods in Numerical Mathematics. 2011;2(1-2):32-38.
31. Naher H, Abdullah FA. Some new traveling wave solutions of the nonlinear reaction diffusion equation by using the improved (G'/G) -expansion method. Mathematical Problems in Engineering, 17 pages, Article ID: 871724, doi:10.1155/2012/871724, 2012.
32. Naher H, Abdullah FA. The improved (G'/G)-expansion method for the (2+1)-dimensional modified Zakharov-Kuznetsov equation. Journal of Applied Mathematics, Article ID: 438928 (2012), 20 pages, doi: 10.1155/2012/438928.
33. Naher H, Abdullah FA, Bekir A. Abundant traveling wave solutions of the compound KdV-Burgers equation via the improved (G'/G)-expansion method. AIP Advances, 2, 042163 (2012); doi: 10.1063/1.4769751.
34. Zhang H. New exact travelling wave solutions for some nonlinear evolution equations. Chaos, Solitons and Fractals. 2005;26:921-925.
35. Wazwaz AM. New travelling wave solutions to the Boussinesq and Klein-Gordon equations. Communications in Nonlinear Science and Numerical Simulation. 2008;13: 889-901.
36. Naher H, Abdullah FA. The basic (G'/G)-expansion method for the fourth order Boussinesq equation. Applied Mathematics. 2012;3(10):144-1152.

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