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Mean-periodic Functions Associated to a Family of Differential-reflection Operators

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The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

We consider the family of differential-reflection operators $\Lambda_{A,\varepsilon}$. We study the harmonic analysis associated with this operator. Next we define and characterize the mean-periodic functions associated with $\Lambda_{A,\varepsilon}$.

Keywords: *Family of differential-reflection operators; generalized mean-periodic functions; Abel summation process.*

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1 Introduction

In this paper, we consider the family of differential-reflection operators on the real line

$$\Lambda f(x) = f'(x) + \frac{A^{'}(x)}{A(x)}\left(\frac{f(x)-f(-x)}{2}\right) - \varepsilon\rho f(-x), \tag{1}$$

where A is a so-called Che'bli function on $\mathbb{R}$, $\rho \geq 0$ is the index of A, and $-1 \leq \varepsilon \leq 1$. Some of our results still hold for arbitrary $\varepsilon \in \mathbb{R}$. However, for simplicity, we will restrict ourselves to the interval $[-1,1]$. The function A and the real number ε are the deformation parameters giving back three well known cases when:

-Dunkl's operators when $A(x) = A_\alpha(x) =\mid x \mid^{2\alpha+1}$ and ε arbitrary.
-Heckman's operators when $A(x) = A_{\alpha,\beta}(x) =\mid sinhx \mid^{2\alpha+1} (coshx)^{2\beta+1}$ and $\varepsilon = 0$.
-Cherednik's operators when $A(x) = A_{\alpha,\beta}(x) =\mid sinhx \mid^{2\alpha+1} (coshx)^{2\beta+1}$ and $\varepsilon = 1$.

Ben said [1] has proved that there exists a unique automorphism of the space $\mathcal{E}(\mathbb{R})$ of C^∞ functions on $\mathbb{R}$, satisfying

$$V_{A,\varepsilon} \circ \frac{d}{dx} f = \Lambda_{A,\varepsilon} \circ V_{A,\varepsilon} f \qquad \text{and} \qquad V_{A,\varepsilon} f(0) = f(0), \tag{2}$$

for all $f \in \mathcal{E}(\mathbb{R})$.

A summary of this harmonic analysis is provided in Sec. 2. Through this paper, the classical theory of mean-periodic functions on $\mathbb{R}$ is extended to the differential-reflection operator $\Lambda_{A,\varepsilon}$. More explicitly, a function f in $\mathcal{E}(\mathbb{R})$ is called $\Lambda_{A,\varepsilon}$-mean-periodic if there exists a non zero compactly supported distribution μ on $\mathbb{R}$, such that

$$\mu\#f(x) = 0, \quad \text{for all } x \in \mathbb{R},$$

$\#$ being the generalized convolution generated by the differential-reflection operator $\Lambda_{A,\varepsilon}$. By using the intertwining operator $V_{A,\varepsilon}$ and the results of Schwartz in the classical setting [2], we express in Sec. 3 the $\Lambda_{A,\varepsilon}$-mean-periodic function f in terms the elementary functions

$$\Phi_{\lambda,l}(x) = V_{A,\varepsilon}\left(y^l e^{i\lambda y}\right)(x).$$

Namely, f may be expanded formally as

$$f(x) = \sum_{(\lambda,l)} \sum_{0\leq s\leq l-1} c_{\lambda,s}\Phi_{\lambda,s}(x), \quad c_{\lambda,s} \in \mathbb{C},$$

the summation being extended over the distinct roots λ of $\mathcal{F}_{A,\varepsilon}(\mu)$ counted with multiplicities l, where $\mathcal{F}_{A,\varepsilon}(\mu)$ stands for the generalized Fourier transform of μ defined by

$$\mathcal{F}_{A,\varepsilon}(\mu)(\lambda) = \langle \mu_y, \Phi_{A,\varepsilon}(-\lambda, y)\rangle, \quad \lambda \in \mathbb{C}.$$

Starting from the distribution μ, we construct in Sec. 4 a biorthogonal system which shows that the coefficients $c_{\lambda,s}$ in the series above are uniquely determined by f. In Sec. 5, we show that the series above is actually convergent to f in the topology of $\mathcal{E}(\mathbb{R})$, after a certain Abel summation procedure is performed. Moreover, we introduce a class of distributions μ for which the Abelian summation process can be dispensed.

In the classical setting, the notion of mean-periodicity was first introduced by Delsarte [3], and then analyzed in depth by Schwartz [1], Kahane [4], Berenstein and Taylor [5]. Later, Trimeche [6] extended the theory of mean-periodic functions to a class of singular second-order differential operator on the half-line . It is pointed out that all the results obtained in theory of mean-periodic function emerge as easy consequences of those stated in the present article.

2 Preliminaries

In this section we provide some facts about harmonic analysis related to the differential-difference operator $\Lambda_{A,\varepsilon}$. We cite here, as briefly as possible, only those properties actually required for the discussion. For more details we refer to [1].

Notation. We denote by

– $\mathcal{E}(\mathbb{R})$ the space of C^∞ functions on $\mathbb{R}$, endowed with the topology of compact convergence for all derivatives;

– $\mathcal{E}^{'}(\mathbb{R})$ the space of distributions on $\mathbb{R}$ with compact support;

– $\mathcal{D}_a(\mathbb{R})$, $a > 0$, the space of C^∞ functions on $\mathbb{R}$ supported in $[-a, a]$, equipped with the topology induced by $\mathcal{E}(\mathbb{R})$;

– $\mathcal{D}(\mathbb{R}) = \cup_{a>0}\mathcal{D}_a(\mathbb{R})$ endowed with the inductive limit topology;

$-PW_a(\mathbb{C})$, be the space of entire functions h on $\mathbb{C}$ wich are of exponential type and rapidly decreasing

$$\exists a > 0, \forall t \in \mathbb{N}, \sup_{\lambda\in\mathbb{C}}(1 + |\lambda|)^t e^{-a|Im\lambda|} \mid h(\lambda) \mid< \infty$$

Remark 2.1. *Clearly $\Lambda_{A,\varepsilon}$ is a bounded linear operator from $\mathcal{E}(\mathbb{R})$ into itself. If $\mu \in \mathcal{E}'(\mathbb{R})$ and $n \in \mathbb{N}$, define $\Lambda^n_{A,\varepsilon}\mu \in \mathcal{E}'(\mathbb{R})$ by*

$$\langle\Lambda^n_{A,\varepsilon}\mu, f\rangle = (-1)^n \langle\mu, \Lambda^n_{A,\varepsilon}f\rangle, \quad f \in \mathcal{E}(\mathbb{R}).$$

2.1 Intertwining operators

Throughout this paper we will denote by A a function on $\mathbb{R}$ satisfying the following:

•$A(x) =| x |^{2\alpha+1} B(x)$, where $\alpha > -\frac{1}{2}$ and B is any even, positive and smooth function on $\mathbb{R}$ with $B(0) = 1$.
•A is increasing and unbounded on $\mathbb{R}_+$.
•$\frac{A'}{A}$ is a decreasing and smooth function on $\mathbb{R}^*_+$, and hence the limit $2\rho := \lim_{x\to+\infty} \frac{A'(x)}{A(x)} \geq 0$ exists.
• There exists a constant $\delta > 0$ such that for all $x \in [x_0, \infty)$ for some$x_0 > 0$,

$$\frac{A'(x)}{A(x)} = \begin{cases} 2\rho + e^{-\rho x}D(x) & \text{if } \rho > 0, \\ \dfrac{2\alpha+1}{x}e^{-\rho x}D(x) \text{ if } \rho = 0, & \end{cases}$$

with D being a smooth function bounded together with its derivatives.

Such a function A is called a Chebli function.

Let $\triangle$, be the following second order differential operator

$$\triangle f(x) = -(\mu^2 + \rho^2)f(x) \text{ with} f(0) = 1 \text{ and} f'(0) = 0. \tag{3}$$

The system 3 admits a unique solution φ_μ. The following Laplace type representation of φ_μ can be found in [7].

For every $x \in \mathbb{R}^*$ there exists a probability measure ν_x on $\mathbb{R}$ supported in $[- \mid x \mid, \mid x \mid]$ such that for all $\mu \in \mathbb{C}$.

$$\varphi_\mu(x) = \int_{-|x|}^{|x|} e^{(i\mu-\rho)t}\nu_x(dt)$$

Also, for $x \in \mathbb{R}^*$, there is a non-negative even continuous function $K(\mid x \mid, .)$ supported in $[- \mid x \mid, \mid x \mid]$ such that for all $\mu \in \mathbb{C}$

$$\varphi_\mu(x) = \int_{-|x|}^{|x|} K(\mid x \mid, t) cos(\mu t) dt.$$

let $\lambda \in \mathbb{C}$ and consider the initial data problem

$$\Lambda_{A,\varepsilon} u = i\lambda u, \qquad \text{with} \qquad u(0) = 1, \tag{4}$$

let $\lambda \in \mathbb{C}$. There exists a unique solution $\Psi_{A,\varepsilon}(\lambda, .)$ to the problem (4). Further, for every $x \in \mathbb{R}$, the function $\lambda \to \Psi_{A,\varepsilon}(\lambda, x)$ is analytic on $\mathbb{C}$. More explicitly:

(i) For $i\lambda \neq \varepsilon\varrho$,

$$\Psi_{A,\varepsilon}(\lambda, x) = \varphi_{\mu_\varepsilon}(x) + \frac{1}{i\lambda - \varepsilon\varrho} \frac{d}{dx} \varphi_{\mu_\varepsilon}(x) \text{ with } \mu_\varepsilon^2 := \lambda^2 + (\varepsilon^2 - 1)\varrho^2. \tag{5}$$

We may rewrite the solution (5) as

$$\Psi_{A,\varepsilon}(\lambda, x) = \varphi_{\mu_\varepsilon}(x) + (i\lambda + \varepsilon\varrho) \frac{sgn(x)}{A(x)} \int_0^{|x|} \varphi_{\mu_\varepsilon}(t) A(t) dt. \tag{6}$$

(ii) For $i\lambda = \varepsilon\varrho$,

$$\Psi_{A,\varepsilon}(\lambda, x) = 1 + 2\varepsilon\varrho \frac{sgn(x)}{A(x)} \int_0^{|x|} A(t) dt. \tag{7}$$

For every $x \in \mathbb{R}^*$, there is a non-negative continuous function $K_\varepsilon(x, .)$ supported in $[- \mid x \mid, \mid x \mid]$ such that for all $\lambda \in \mathbb{C}$,

$$\Psi_{A,\varepsilon}(\lambda, x) = \int_{|y|<|x|} K_\varepsilon(x, y) e^{i\lambda y} dy. \tag{8}$$

For $f \in \mathcal{E}(\mathbb{R})$ we define $V_{A,\varepsilon} f$ by

$$V_{A,\varepsilon} f(x) = \int_{|y|<|x|} K_\varepsilon(x, y) f(y) dy \,\text{for}\, x \neq 0, \text{ and } V_{A,\varepsilon} f(0) = f(0).$$

Observe that

$$\Psi_{A,\varepsilon}(\lambda, x) = V_{A,\varepsilon}(e^{i\lambda .})(x). \tag{9}$$

Theorem 2.1. [*1*] *The operator $V_{A,\varepsilon}$ is the unique automorphism of $\mathcal{E}(\mathbb{R})$ such that*

$$\Lambda_{A,\varepsilon} \circ V_{A,\varepsilon} = V_{A,\varepsilon} \circ \frac{d}{dx} \tag{10}$$

where $\Lambda_{A,\varepsilon}$ is the family of differential-reflection operator.

Below we will deal with the dual operator ${}^t V_{A,\varepsilon}$ of $V_{A,\varepsilon}$ in the sense that

$$\int_{\mathbb{R}} V_{A,\varepsilon} f(x) g(x) A(x) dx = \int_{\mathbb{R}} f(y) {}^t V_{A,\varepsilon} g(y) dy$$

This can be written

$${}^t V_{A,\varepsilon} g(y) = \int_{|y|<|x|} K_\varepsilon(x, y) g(x) A(x) dx.$$

Theorem 2.2. *The integral transform ${}^t V_{A,\varepsilon}$ is a topological automorphism of $\mathcal{D}(\mathbb{R})$ satisfying the intertwining relation*

$$\frac{d}{dy} {}^t V_{A,\varepsilon} f = {}^t V_{A,\varepsilon} (\Lambda_{A,\varepsilon} + 2\varepsilon\varrho S), \quad f \in \mathcal{D}(\mathbb{R}),$$

where S denotes the symmetry $(Sf)(x) := f(-x)$.

For more details you can see [1].

2.2 Generalized fourier transform

The generalized Fourier transform of a distribution $\mu \in \mathcal{E}^{'}(\mathbb{R})$ is defined by

$$\mathcal{F}_{A,\varepsilon}(\mu)(\lambda) = \langle \mu, \Phi_{A,\varepsilon}(-\lambda, .)\rangle , \quad \lambda \in \mathbb{C}.$$

Assume that $-1 \leq \varepsilon \leq 1$. For $f \in \mathcal{D}(\mathbb{R})$ The generalized Fourier transform is defined by

$$\mathcal{F}_{A,\varepsilon}(f)(\lambda) = \int_{\mathbb{R}} f(x)\Psi_{A,\varepsilon}(\lambda, -x)A(x)dx, \quad \lambda \in \mathbb{C}.$$

Recall the following identities :

$$\mathcal{F}_{A,\varepsilon}(\mu) = \mathcal{F}_u\left({}^tV_{A,\varepsilon}\mu\right), \quad \mu \in \mathcal{E}^{'}(\mathbb{R}), \tag{11}$$

$$\mathcal{F}_{A,\varepsilon}(f) = \mathcal{F}_u\left({}^tV_{A,\varepsilon}f\right), \quad f \in \mathcal{D}(\mathbb{R}),$$

$$\mathcal{F}_{A,\varepsilon}(\Lambda_{A,\varepsilon}\mu)(\lambda) = i\lambda\, \mathcal{F}_{A,\varepsilon}(\mu)(\lambda), \quad \mu \in \mathcal{E}^{'}(\mathbb{R}),$$

$$\mathcal{F}_{A,\varepsilon}(\Lambda_{A,\varepsilon} + 2\varepsilon\varrho S)(\lambda) = i\lambda\, \mathcal{F}_{\Lambda}(f)(\lambda), \quad f \in \mathcal{D}(\mathbb{R}),$$

$\mathcal{F}_u$ being the usual Fourier transform on $\mathbb{R}$ given by

$$\mathcal{F}_u(\mu)(\lambda) = \int_{\mathbb{R}} e^{-i\lambda x}\, d\mu(x), \quad \mu \in \mathcal{E}^{'}(\mathbb{R}).$$

An outstanding result about the generalized Fourier transform $\mathcal{F}_{A,\varepsilon}$ is as follows.

Theorem 2.3. ***(Paley-Wiener)***

(i) The generalized Fourier transform $\mathcal{F}_{A,\varepsilon}$ is a bijection from $\mathcal{E}^{'}(\mathbb{R})$ onto $PW(\mathbb{C})$. More precisely, μ has its support in $[-a, a]$ if, and only if, $\mathcal{F}_{A,\varepsilon}(\mu) \in \mathcal{H}_a$.

(ii) The generalized Fourier transform $\mathcal{F}_{A,\varepsilon}$ is a topological isomorphism from $\mathcal{D}(\mathbb{R})$ onto $PW(\mathbb{C})$. More precisely, $f \in \mathcal{D}_a(\mathbb{R})$ if, and only if, $\mathcal{F}_{A,\varepsilon}(f) \in PW_a(\mathbb{C})$.

2.3 Generalized convolution

The generalized translation operators T^x, $x \in \mathbb{R}$, tied to $\Lambda_{A,\varepsilon}$ are defined on $\mathcal{E}(\mathbb{R})$ by

$$T^x f(y) = V_{A,\varepsilon,x}V_{A,\varepsilon,y}\left[V^{-1}_{A,\varepsilon}f(x+y)\right], \quad y \in \mathbb{R}.$$

The T^x, $x \in \mathbb{R}$, are linear bounded operator from $\mathcal{E}(\mathbb{R})$ into itself, and possess the following fundamental properties:

$$T^0 = \text{identity}, \quad T^xT^y = T^yT^x, \quad T^xf(y) = T^yf(x),$$

$$\Lambda_{A,\varepsilon}T^x = T^x\Lambda_{A,\varepsilon} \quad \text{and} \quad (T^x\Psi_{A,\varepsilon})(\lambda, y) = \Psi_{A,\varepsilon}(\lambda, x)\Psi_{A,\varepsilon}(\lambda, y).$$

The generalized convolution product of two distributions $\mu, \nu \in \mathcal{E}^{'}(\mathbb{R})$, is the distribution $\mu\#\nu \in \mathcal{E}^{'}(\mathbb{R})$ given by

$$\langle \mu\#\nu, f\rangle = \langle \mu_x, \langle \nu_y, T^xf(y)\rangle\rangle , \quad f \in \mathcal{E}(\mathbb{R}).$$

The generalized convolution of $\mu \in \mathcal{E}^{'}(\mathbb{R})$ and $f \in \mathcal{E}(\mathbb{R})$, is the function $\mu\#f \in \mathcal{E}(\mathbb{R})$ given by

$$\mu\#f(x) = \left\langle \mu_y, T^{-x}f^-(y)\right\rangle , \quad x \in \mathbb{R},$$

with $f^-(y) = f(-y)$.

Proposition 2.1. *(i) Let* $\mu, \nu \in \mathcal{E}^{'}(\mathbb{R})$ *and* $f \in \mathcal{D}(\mathbb{R})$*. Then*

$$\mathcal{F}_{A,\varepsilon}(\mu\#\nu) = \mathcal{F}_{A,\varepsilon}(\mu)\mathcal{F}_{A,\varepsilon}(\nu), \tag{12}$$

$$\mathcal{F}_{A,\varepsilon}(\mu\#f) = \mathcal{F}_{A,\varepsilon}(\mu)\mathcal{F}_{A,\varepsilon}(f). \tag{13}$$

(ii) For $\mu, \nu \in \mathcal{E}^{'}(\mathbb{R})$ *and* $f \in \mathcal{E}(\mathbb{R})$ *we have*

$$\mu\#(\nu\#f) = (\mu\#\nu)\#f.$$

(iii) If $\mu, \nu \in \mathcal{E}^{'}(\mathbb{R})$ *and* $f \in \mathcal{E}(\mathbb{R})$ *then*

$$V_{A,\varepsilon}\left({}^{t}V_{A,\varepsilon}\mu * f\right) = \mu \# V_{A,\varepsilon}f, \tag{14}$$

$${}^{t}V_{A,\varepsilon}(\mu\#\nu) = {}^{t}V_{A,\varepsilon}\mu * {}^{t}V_{A,\varepsilon}\nu,$$

where $*$ *denotes the classical convolution on* $\mathbb{R}$.

3 $\Lambda_{A,\varepsilon}$-mean-periodic Functions

According to Schwartz [2], a function f in $\mathcal{E}(\mathbb{R})$ is called mean-periodic relatively to a distribution μ in $\mathcal{E}^{'}(\mathbb{R})$, if it is a solution of the convolution equation

$$\mu * f(x) = 0, \quad \text{for all } x \in \mathbb{R}.$$

In this section we extend the notion of mean-periodicity to the differential-difference operator $\Lambda_{A,\varepsilon}$, by replacing in the equation above the ordinary convolution * by the generalized convolution #.

Definition 3.1. *We say that a function* $f \in \mathcal{E}(\mathbb{R})$ *is* $\Lambda_{A,\varepsilon}$*-mean-periodic, if there exists* $0 \neq \mu \in \mathcal{E}^{'}(\mathbb{R})$ *such that*

$$\mu\#f(x) = 0, \quad \text{for all } x \in \mathbb{R}.$$

If we want to emphasize the equation satisfied by f *we will say that* f *is mean-periodic with respect to* μ *or* μ*-*$\Lambda_{A,\varepsilon}$*-mean-periodic.*

Notation. For $f \in \mathcal{E}(\mathbb{R})$, write $\tau(f)$ for the closure of the subspace of $\mathcal{E}(\mathbb{R})$ spanned by $T^{-x}f^{-},\ x \in \mathbb{R}$.

Remark 3.1. *(i) Notice that*

$$\mu\#f = 0 \Leftrightarrow \mu = 0 \text{ on } \tau(f) \Leftrightarrow \mu \in (\tau(f))^{\perp}$$

(ii) According to the Hahn-Banach theorem, Definition 3.1 is equivalent to $\tau(f) \neq \mathcal{E}(\mathbb{R})$.

Examples. (i) Let a be a nonzero real number. Each function $f \in \mathcal{E}(\mathbb{R})$ satisfying

$$T^{-x}f^{-}(a) = f(x), \quad \text{for all } x \in \mathbb{R},$$

is $\Lambda_{A,\varepsilon}$-mean-periodic with respect to $\mu = \delta_a - \delta_0$, where δ_a denotes the Dirac measure at the point a.

(ii) By virtue of (6) and Theorem 2.2, every $0 \neq f \in \mathcal{D}(\mathbb{R})$ is not $\Lambda_{A,\varepsilon}$-mean-periodic.

Proposition 3.1. *For* $\lambda \in \mathbb{C}$*,* $x \in \mathbb{R}$ *and* $l \in \mathbb{N}$*, put*

$$\phi_{\lambda,l}(x) = x^{l}e^{i\lambda x} \quad \text{and} \quad \Phi_{\lambda,l}(x) = V_{A,\varepsilon}(\phi_{\lambda,l})(x) \tag{15}$$

Then

(i) $\Phi_{\lambda,l}(x) = (-i)^l \dfrac{\partial^l}{\partial \lambda^l} \Psi_{A,\varepsilon}(\lambda, x)$.

(ii) For all $\mu \in \mathcal{E}'(\mathbb{R})$*, we have*

$$(\mathcal{F}_{A,\varepsilon}(\mu))^{(l)}(\lambda) = (-i)^l \langle \mu, \Phi_{-\lambda,l} \rangle, \tag{16}$$

$$\mu \# \Phi_{\lambda,l}(x) = \sum_{s=0}^{l} \begin{pmatrix} l \\ s \end{pmatrix} \Phi_{\lambda,l-s}(x)(-i)^s (\mathcal{F}_{\Lambda}(\mu))^{(s)}(\lambda). \tag{17}$$

(iii) The function $x \to \Phi_{\lambda,l}(x)$ *is* $\Lambda_{A,\varepsilon}$*-mean-periodic.*

Proof. Assertion (i) :

$$\begin{aligned} \Phi_{\lambda,l}(x) &= V_{A,\varepsilon}(\phi_{\lambda,l}(x)) \\ &= \textstyle\int_{|y|<|x|} K_\varepsilon(x,y)\phi_{\lambda,l}(y)dy \\ &= \textstyle\int_{|y|<|x|} K_\varepsilon(x,y)(y^l e^{i\lambda y})dy \\ &= \textstyle\int_{|y|<|x|} K_\varepsilon(x,y)(-i)^l \dfrac{\partial^l}{\partial \lambda^l}(e^{i\lambda y})dy \\ &= (-i)^l \textstyle\int_{|y|<|x|} K_\varepsilon(x,y) \dfrac{\partial^l}{\partial \lambda^l}(e^{i\lambda y})dy, \end{aligned}$$

using Leibniz Rule we get,

$$\begin{aligned} \Phi_{\lambda,l}(x) &= (-i)^l \frac{\partial^l}{\partial \lambda^l}\left(\int_{|y|<|x|} K_\varepsilon(x,y)e^{i\lambda y}dy\right) \\ &= (-i)^l \frac{\partial^l}{\partial \lambda^l} \Psi_{A,\varepsilon}(\lambda, x). \end{aligned}$$

Formula (16) follows also by using differentiation under the integral sign. Let us check (17). By (15) and (14),

$$\mu \# \Phi_{\lambda,l} = V_{A,\varepsilon}\left({}^t V_{A,\varepsilon}\mu * \phi_{\lambda,l}\right). \tag{18}$$

But an easy computation shows that

$$\nu * \phi_{\lambda,l}(x) = \sum_{s=0}^{l} \begin{pmatrix} l \\ s \end{pmatrix} \phi_{\lambda,l-s}(x)(-i)^s (\mathcal{F}_u(\nu))^{(s)}(\lambda),$$

for all $\nu \in \mathcal{E}'(\mathbb{R})$. So

$${}^t V_{A,\varepsilon}\mu * \phi_{\lambda,l}(x) = \sum_{s=0}^{l} \begin{pmatrix} l \\ s \end{pmatrix} \phi_{\lambda,l-s}(x)(-i)^s (\mathcal{F}_{A,\varepsilon}(\nu))^{(s)}(\lambda), \tag{19}$$

by virtue of (11). Identity (17) follows now by combining (15), (18) and (19). Finally, to have $\mu \# \Phi_{\lambda,l} \equiv 0$, it is sufficient in view of (17), to choose $0 \neq \mu \in \mathcal{E}'(\mathbb{R})$ such that λ is a zero of order at least l of $\mathcal{F}_{A,\varepsilon}(\mu)$. This completes the proof. □

Proposition 3.2. *Let* $f \in \mathcal{E}(\mathbb{R})$ *be* $\Lambda_{A,\varepsilon}$*-mean-periodic. Then* $\Phi_{\lambda,l} \in \tau(f)$ *if and only if, for all* $\mu \in (\tau(f))^{\perp}$*, we have*

$$(\mathcal{F}_{A,\varepsilon}(\mu))^{(l)}(-\lambda) = 0.$$

Proof. The result follows by using (16) and the Hahn-Banach theorem. □

Definition 3.2. *We call spectrum of a $\Lambda_{A,\varepsilon}$-mean-periodic function $f \in \mathcal{E}(\mathbb{R})$, denoted by $sp(f)$, the set of pairs (λ, l), $\lambda \in \mathbb{C}$, $l \in \mathbb{N}$, such that the functions $\Phi_{\lambda,s}$ belong to $\tau(f)$ for $0 \leq s \leq l-1$ and not for $s = l$.*

Remark 3.2. *According to Proposition* 3.2*, $(\lambda, l) \in sp(f)$ if and only if, $-\lambda$ is a common zero of order l of the generalized Fourier transforms of elements of $(\tau(f))^{\perp}$.*

The next statement clarifies the relationship between $\Lambda_{A,\varepsilon}$-mean-periodic functions and classical mean-periodic functions.

Proposition 3.3. *A function $f \in \mathcal{E}(\mathbb{R})$ is $\Lambda_{A,\varepsilon}$-mean-periodic with respect to a distribution $\mu \in \mathcal{E}'(\mathbb{R})$ if, and only if, $V_{A,\varepsilon}^{-1} f$ is a classical mean-periodic function with respect to ${}^{t}V_{A,\varepsilon}\mu$.*

Proof. The result is a direct consequence of (14). □

From the work of Schwartz [2] and the proposition above, we deduce the following characterization of Λ-mean-periodic functions.

Theorem 3.1. *Let $f \in \mathcal{E}(\mathbb{R})$ be $\Lambda_{A,\varepsilon}$-mean-periodic. Then f can be approximated in the topology of $\mathcal{E}(\mathbb{R})$ by finite linear combinations of functions of the type $\Phi_{\lambda,l}$, $(\lambda, l) \in sp(f)$.*

4 Biorthogonal System

Notation. Throughout this section fix $0 \neq \mu \in \mathcal{E}'(\mathbb{R})$. Put

$$\mathcal{Z}_{A,\varepsilon}(\mu) = \{(\lambda_k, l_k), k \in \mathbb{N}, l_k \in \mathbb{N}\},$$

where λ_k is a zero of order l_k of the entire function $\mathcal{F}_{A,\varepsilon}(\mu)$.

Starting from the distribution μ, we construct in this section a biorthogonal system in $\mathcal{E}'(\mathbb{R})$, that is, a family of distributions $\mu_{k,m} \in \mathcal{E}'(\mathbb{R})$, satisfying

$$\langle \mu_{k,m}, \Phi_{\lambda_s,j} \rangle = \delta_{k,s}\, \delta_{m,j} \tag{20}$$

for $0 \leq m \leq l_k - 1$ and $0 \leq j \leq l_s - 1$. Given a μ-$\Lambda_{A,\varepsilon}$-mean-periodic function $f \in \mathcal{E}(\mathbb{R})$, formula (20) will allow us to compute the coefficients $c_{k,l}$ in a possible development of f with respect to the functions $\Phi_{\lambda_k,l}$, $k \in \mathbb{N}$, $0 \leq l \leq l_k - 1$. We adopt here the arguments used by Delsarte [3] and Schwartz [2].

Notation. For $f \in \mathcal{E}(\mathbb{R})$, put

$$I_k(f)(x) = \int_0^x f(t) e^{i\lambda_k(x-t)} dt, \quad x \in \mathbb{R}.$$

Lemma 4.1. *Let $f \in \mathcal{E}(\mathbb{R})$. Then*

(i) The general solution of the equation

$$\left(\frac{d}{dx} - i\lambda_k\right)^{l_k} g = f,$$

is given by

$$g(x) = \sum_{s=0}^{l_k-1} \beta_s\, \phi_{\lambda_k,s}(x) + \overbrace{I_k \circ \cdots \circ I_k}^{l_k\, times}(f)(x), \quad \beta_s \in \mathbb{C}.$$

(ii) The general solution of the equation

$$(\Lambda_{A,\varepsilon} - i\lambda_k)^{l_k}\, g = f, \tag{21}$$

is given by

$$g(x) = \sum_{s=0}^{l_k-1} \beta_s\, \Phi_{\lambda_k,s}(x) \,+\, V_{A,\varepsilon} \circ \overbrace{I_k \circ \cdots \circ I_k}^{l_k\ times} \circ V_{A,\varepsilon}^{-1}(f)(x), \quad \beta_s \in \mathbb{C}.$$

Proof. Assertion (i) is easily checked. By virtue of (10), equation (21) is equivalent to

$$\left(\frac{d}{dx} - i\lambda_k\right)^{l_k} \left(V_{A,\varepsilon}^{-1} g\right) = V_{A,\varepsilon}^{-1} f.$$

Assertion (ii) follows then from (i). □

Lemma 4.2. *There is a unique distribution $\mu_- \in \mathcal{E}'(\mathbb{R})$ such that*

$$\mathcal{F}_{A,\varepsilon}(\mu_-)(\lambda) = \mathcal{F}_{A,\varepsilon}(\mu)(-\lambda), \quad \text{for all } \lambda \in \mathbb{C}.$$

Moreover, if $\operatorname{supp}\mu \subset [-a,a]$*, then* $\operatorname{supp}(\mu_-) \subset [-a,a]$.

Proof. The result follows readily from Theorem 2.2(i). □

Remark 4.1. *Define $\mu^- \in \mathcal{E}'(\mathbb{R})$ by*

$$\int_{\mathbb{R}} f(x) d\mu^-(x) = \int_{\mathbb{R}} f(-x) d\mu(x), \quad f \in \mathcal{E}(\mathbb{R}).$$

Then according to (6) *and Theorem* 2.2*(i), $\mu_- = \mu^-$ if and only if $\rho = 0$.*

Notation. If G is a meromorphic function, having γ as a pole, we denote by $[G(\lambda)]_\gamma$ the singular part of $G(\lambda)$ in a neighborhood of γ, hence $G(\lambda) - [G(\lambda)]_\gamma$ is holomorphic in a neighborhood of γ.

Lemma 4.3. *(i) The distribution $q_k \in \mathcal{E}'(\mathbb{R})$ defined by*

$$\mathcal{F}_{A,\varepsilon}(q_k)(\lambda) = (\lambda + \lambda_k)^{l_k} \left[\frac{1}{\mathcal{F}_{A,\varepsilon}(\mu)(-\lambda)}\right]_{-\lambda_k}$$

has a support concentrated at the origin.

(ii) The distribution $\mu_{k,0} \in \mathcal{E}'(\mathbb{R})$ defined by

$$\mathcal{F}_{A,\varepsilon}(\mu_{k,0})(\lambda) = \begin{cases} \mathcal{F}_{A,\varepsilon}(\mu)(-\lambda) \left[\frac{1}{\mathcal{F}_{A,\varepsilon}(\mu)(-\lambda)}\right]_{-\lambda_k} & \text{if } \lambda \neq -\lambda_k, \\ 1 & \text{if } \lambda = -\lambda_k, \end{cases} \tag{22}$$

satisfies

$$\langle \mu_{k,0}, f\rangle = (-i)^{l_k} \left\langle q_k \# \mu_-, V_{A,\varepsilon} \circ \overbrace{I_k \circ \cdots \circ I_k}^{l_k\ times} \circ V_{A,\varepsilon}^{-1}(f) \right\rangle,$$

for all $f \in \mathcal{E}(\mathbb{R})$.

Proof. (i) As the function $(\lambda+\lambda_k)^{l_k}\left[1/\mathcal{F}_{A,\varepsilon}(\mu)(-\lambda)\right]_{-\lambda_k}$ is a polynomial $P_k(\lambda)$, it follows by (11) that ${}^tV_{A,\varepsilon}q_k = P_k(d/dx)(\delta_0)$. Then using Theorem 2.1(i), we deuce that q_k has a support concentrated at the origin.

(ii) As

$$(\lambda+\lambda_k)^{l_k}\,\mathcal{F}_{A,\varepsilon}(\mu_{k,0})(\lambda) = \mathcal{F}_{A,\varepsilon}(q_k)(\lambda)\mathcal{F}_{A,\varepsilon}(\mu)(-\lambda),$$

it follows from (12) that

$$(-i)^{l_k}\,(\Lambda_{A,\varepsilon}+i\lambda_k)^{l_k}\,\mu_{k,0} = q_k\#\mu_-.$$

So for all g in $\mathcal{E}(\mathbb{R})$,

$$\langle q_k\#\mu_-, g\rangle = (-i)^{l_k}\left\langle (\Lambda_{A,\varepsilon}+i\lambda_k)^{l_k}\mu_{k,0}, g\right\rangle = i^{\,l_k}\left\langle \mu_{k,0}, (\Lambda_{A,\varepsilon}-i\lambda_k)^{l_k} g\right\rangle.$$

The result is now a direct consequence of (16) and Lemma 4.1(ii). □

Remark 4.2. *If the zeros λ_k of $\mathcal{F}_{A,\varepsilon}(\mu)$ are simple, then*

$$\left[\frac{1}{\mathcal{F}_{A,\varepsilon}(\mu)(-\lambda)}\right]_{-\lambda_k} = \frac{-1}{(\lambda+\lambda_k)\,(\mathcal{F}_{A,\varepsilon}(\mu))'(-\lambda_k)},$$

that is,

$$q_k = \frac{-\,\delta_0}{(\mathcal{F}_{A,\varepsilon}(\mu))'(-\lambda_k)}$$

and

$$\langle \mu_{k,0}, f\rangle = \frac{i}{(\mathcal{F}_{A,\varepsilon}(\mu))'(-\lambda_k)}\,\langle \mu_-, V_{A,\varepsilon}\circ I_k\circ V_{A,\varepsilon}^{-1}(f)\rangle$$

for all $f\in\mathcal{E}(\mathbb{R})$.

Proposition 4.1. *Define $\mu_{k,m}\in\mathcal{E}'(\mathbb{R})$, $0\le m\le l_k-1$, by*

$$\mu_{k,m} = \frac{(-1)^m}{m!}\,(\Lambda_{A,\varepsilon}+i\lambda_k)^m\,\mu_{k,0} + \tau_{k,m}\,\#\mu_-, \tag{23}$$

where

– $\mu_{k,0}\in\mathcal{E}'(\mathbb{R})$ *is defined in Lemma 4.3.*

– $\tau_{k,m}\in\mathcal{E}'(\mathbb{R})$ *with support concentrated at the origin, whose the generalized Fourier transform is given by*

$$\mathcal{F}_{A,\varepsilon}(\tau_{k,m})(\lambda) = \frac{(-i)^m}{m!}\,R_{k,m}(\lambda) \tag{24}$$

with

$$R_{k,m}(\lambda) = \left[\frac{(\lambda+\lambda_k)^m}{\mathcal{F}_{A,\varepsilon}(\mu)(-\lambda)}\right]_{-\lambda_k} - (\lambda+\lambda_k)^m\left[\frac{1}{\mathcal{F}_{A,\varepsilon}(\mu)(-\lambda)}\right]_{-\lambda_k}$$

Then the family $(\mu_{k,m})$ satisfies (20).

Proof. Notice that $R_{k,m}(\lambda)$ is a polynomial, so the support of $\tau_{k,m}$ is concentrated at the origin. A combination of (22), (23) and (24) yields

$$\mathcal{F}_{A,\varepsilon}(\mu_{k,m})(\lambda) = \frac{(-i)^m}{m!}\,\mathcal{F}_{A,\varepsilon}(\mu)(-\lambda)\left[\frac{(\lambda+\lambda_k)^m}{\mathcal{F}_{A,\varepsilon}(\mu)(-\lambda)}\right]_{-\lambda_k}. \tag{25}$$

According to (16) and (25), $\langle \mu_{k,m}, \Phi_{\lambda_s,j}\rangle = 0$ for $s\neq k$. A straightforward calculation shows that

$$\mathcal{F}_{A,\varepsilon}(\mu_{k,m})(\lambda) = (-i)^m\,\frac{(\lambda+\lambda_k)^m}{m!} + O\left((\lambda+\lambda_k)^{l_k+1}\right),$$

in a neighborhood of $-\lambda_k$. We conclude, in view of (16), that $\langle \mu_{k,m}, \Phi_{\lambda_k,j}\rangle = 0$ for $j\neq m$, and $\langle \mu_{k,m}, \Phi_{\lambda_k,m}\rangle = 1$. This achieves the proof. □

Corollary 4.1. *Let $f \in \mathcal{E}(\mathbb{R})$. Assume that there are disjoint finite subsets $\mathcal{Z}_j$ (groupings) such that $\mathcal{Z}_{A,\varepsilon}(\mu) = \bigcup_1^\infty \mathcal{Z}_j$ and*

$$\sum_{j=1}^{\infty} \left(\sum_{(\lambda_k, l_k) \in \mathcal{Z}_j} \sum_{0 \le l \le l_k - 1} c_{k,l} \, \Phi_{\lambda_k, l} \right) \tag{26}$$

is convergent in $\mathcal{E}(\mathbb{R})$ to f, with a suitable mode of convergence. Then f is μ-$\Lambda_{A,\varepsilon}$-mean-periodic and the coefficients $c_{k,l}$ can be computed by the formula

$$c_{k,l} = \langle \mu_{k,l}, f \rangle \, . \tag{27}$$

Proof. The function f is μ-$\Lambda_{A,\varepsilon}$-mean-periodic because that is true for each term in (26). Identity (27) follows immediately from Proposition 4.1. □

5 Series Expansion with Respect to the Functions Φ_{λ_k, l_k}

Like in the classical setting, the series (26) is not actually convergent in $\mathcal{E}(\mathbb{R})$, without a certain abelian summation procedure is performed :

Theorem 5.1. *Let $f \in \mathcal{E}(\mathbb{R})$ be $\Lambda_{A,\varepsilon}$-mean-periodic with respect to $\mu \in \mathcal{E}'(\mathbb{R})$. Then there are disjoint finite subsets $\mathcal{Z}_j$ (groupings) such that $\mathcal{Z}_{A,\varepsilon}(\mu) = \bigcup_1^\infty \mathcal{Z}_j$ and for every $\varepsilon > 0$ the series*

$$\sum_{j=1}^{\infty} \left(\sum_{(\lambda_k, l_k) \in \mathcal{Z}_j} \sum_{0 \le l \le l_k - 1} c_{k,l} \, \Phi_{\lambda_k, l} \, e^{-\varepsilon |\lambda_k|} \right)$$

converges in $\mathcal{E}(\mathbb{R})$ to a function f_ε satisfying :

$$\lim_{\varepsilon \longrightarrow 0} f_\varepsilon = f, \quad \text{in } \mathcal{E}(\mathbb{R}).$$

The coefficients $c_{k,l}$ being determined by (27).

Proof. By Proposition 3.3, $V_{A,\varepsilon}^{-1} f$ is a classical mean-periodic function with respect to the distribution ${}^t V_{A,\varepsilon}\mu$. So using (11) and the results of Schwartz [2], we can find:

- finite subsets $\mathcal{Z}_j$ such that $\mathcal{Z}_{A,\varepsilon}(\mu) = \bigcup_1^\infty \mathcal{Z}_j$
- a sequence of complex numbers $\tilde{c}_{k,l}$

such that for every $\varepsilon > 0$ the series

$$\sum_{j=1}^{\infty} \left(\sum_{(\lambda_k, l_k) \in \mathcal{Z}_j} \sum_{0 \le l \le l_k - 1} \tilde{c}_{k,l} \, \phi_{\lambda_k, l} \, e^{-\varepsilon |\lambda_k|} \right)$$

converges in $\mathcal{E}(\mathbb{R})$ to a function f_ε satisfying :

$$\lim_{\varepsilon \longrightarrow 0} f_\varepsilon = V^{-1} f, \quad \text{in } \mathcal{E}(\mathbb{R}).$$

As the intertwining operator $V_{A,\varepsilon}$ is an automorphism of $\mathcal{E}(\mathbb{R})$, it follows by (11) that

$$V_{A,\varepsilon}(f_\varepsilon) = \sum_{j=1}^{\infty} \left(\sum_{(\lambda_k, l_k) \in \mathcal{Z}_j} \sum_{0 \le l \le l_k - 1} \tilde{c}_{k,l} \, \Phi_{\lambda_k, l} \, e^{-\varepsilon |\lambda_k|} \right)$$

and

$$\lim_{\varepsilon \to 0} V_{A,\varepsilon}(f_\varepsilon) = f,$$

where both the series and the limit are meaningful in the topology of $\mathcal{E}(\mathbb{R})$. Finally, we deduce from Corollary 4.1 that

$$\tilde{c}_{k,l} = c_{k,l}, \qquad 0 \leq l \leq l_k - 1, \quad k \in \mathbb{N}.$$

This ends the proof. □

Following Ehrenpreis [8], we introduce a class of distributions for which the Abel summation process is not necessary.

Definition 5.1. *A distribution $\mu \in \mathcal{E}^{'}(\mathbb{R})$ is called $\Lambda_{A,\varepsilon}$-slowly-decreasing, if there are positive constants c, d such that for any $x \in \mathbb{R}$,*

$$\max\left\{|\mathcal{F}_{A,\varepsilon}(\mu)(y)|,\, y \in \mathbb{R},\, |x-y| \leq d \log\left(1+|x|^2\right)\right\} \geq c\,(1+|x|)^{-1/c}.$$

Using the results of [8] and Proposition 3.3, it is not hard to establish the following theorem.

Theorem 5.2. *Let $f \in \mathcal{E}(\mathbb{R})$ be $\Lambda_{A,\varepsilon}$-mean-periodic with respect to a Λ-slowly-decreasing distribution $\mu \in \mathcal{E}'(\mathbb{R})$. Then there exist finite groupings $\mathcal{Z}_j$ of $\mathcal{Z}_{A,\varepsilon}(\mu)$ such that the series*

$$\sum_{j=1}^{\infty}\left(\sum_{(\lambda_k,l_k)\in\mathcal{Z}_j}\ \sum_{0\leq l\leq l_k-1} c_{k,l}\,\Phi_{\lambda_k,l}\right) \tag{28}$$

converges to f in $\mathcal{E}(\mathbb{R})$. The coefficients $c_{k,l}$ being determined by (27).

The next statement characterizes the Λ-slowly-decreasing distributions $\mu \in \mathcal{E}'(\mathbb{R})$ for which every grouping $\mathcal{Z}_j$ in (28) can be taken to contain a single point of $\mathcal{Z}_{A,\varepsilon}(\mu)$.

Theorem 5.3. *Let $f \in \mathcal{E}(\mathbb{R})$ be $\Lambda_{A,\varepsilon}$-mean-periodic with respect to a Λ-slowly-decreasing distribution $\mu \in \mathcal{E}'(\mathbb{R})$. A necessary and sufficient condition that the series* (28) *converges to f in $\mathcal{E}(\mathbb{R})$ without groupings (i.e.,* $\mathrm{card}(\mathcal{Z}_j) = 1$ *for all j) is that for some c, $d > 0$ we have*

$$\left|\frac{d^l}{d\lambda^l}\mathcal{F}_{A,\varepsilon}(\mu)(\lambda)\right| \geq d\,\frac{\exp(-\,c\,|\mathcal{I}m\lambda|)}{(1+|\lambda|)^c}$$

for all $(\lambda, l) \in \mathcal{Z}_{A,\varepsilon}(\mu)$.

Proof. The result follows easily by combining the results of [5] and Proposition 3.3. □

Competing Interests

Author has declared that no competing interests exist.

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