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Implicit Hybrid Points Approach For Solving General Second Order Ordinary Differential Equations With Initial Values

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Authors' contributions

This work was carried out in collaboration among the authors. The three authors read and approve the final manuscript.

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Abstract

This article focuses on the development of implicit hybrid method for the general solution of second order ordinary differential equations with initial values (IVPs). The method of collocation and interpolation of power series was used to derive the method, while Taylor series is used to develop and analyze the predictors y_{n+i} and y'_{n+i} $i = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$. The method is found to be consistent and zero-stable. The method shows to be more efficient and accurate when compared with existing work by other authors.

Keywords: Power series; Taylor's series; implicit method; initial values problems (IVPs); first order ordinary differential equations (ODEs).

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1 Introduction

Differential equations are often being encountered in sciences, social sciences and engineering. This article focuses on general second order ordinary differential equation with initial value problems of this form:

$$y'' = f(x, y, y'), y(a) = y_0, y'(a) = \beta \tag{1}$$

Customarily, second order ordinary differential equations are solved by reducing it to a system of first order ODEs and then appropriate numerical method for first order is used to solve the resulting system.

The reduction process has been discussed by various authors such as Lambert [1], Fatunla [2], Brugnano and Trigiante [3] and Awoyemi [4]. The approach was very successful but with some drawbacks as discussed by Onumanyi et al. [5], Awoyemi [4], Awoyemi and Kayode [6], Adesanya et al. [7] and Badmus and Yahaya [8]. These drawbacks are computer programs associated with the methods which are mostly complicated when incorporating subroutines to supply the starting values for the methods which invariably resulted into more computer time and more computational burden. Many authors have developed methods for the direct solution of (1) without reducing it to systems of first order ordinary differential equations. Linear multistep method (LMM) for the direct solution of (1) have been considered by Brown [9], Lambert [1], Awoyemi [4, 10,11], Adesanya et al. [7]. They independently proposed linear multistep methods with continuous coefficients to solve (1) in the predictor-corrector and block mode based on collocation and interpolation method and used Taylor's series expansion to supply starting values. Hybrid block method for the solution of third order ordinary differential equations was carried out by Ogunware et al. [12]. Bolarinwa et al. [13] proposed Taylor series approximation method to improve on the setback usually faced with Predictor-Corrector and Block methods. Olanegan et al. [14] developed continuous hybrid linear multistep method (CHLMM) of one-step for the generalized solution of second order ordinary differential equations. The work extended the results generated to solve second order ordinary differential equation by multistep collocation and interpolation technique using Taylor series for implementation. This method helps to investigate the impact of the interpolation point which on substitution and evaluation obtained the direct integration of (1) without reduction to systems of first order differential equations.

In this article, we developed a two-point continuous hybrid method of better accuracy to approximate (1) directly with the use of Taylor series for implementation and evaluation.

We tested our method on some application questions of second order IVPs ordinary differential equation and compared our result with existing methods.

2 The Method

In this section, we apply the interpolation and collocation procedures and we choose our interpolation at the first two point of the method and our collocation point at both grid and off-grid points.

We consider a power series in the form:

$$y(x) = \sum_{j=0}^{(i+c)} a_j x^j \tag{2}$$

Where i and c are the number of the interpolation and collocation points respectively.

The first and second derivatives are

$$y'(x) = \sum_{j=1}^{(i+1)-1} ja_j x^{j-1} \tag{3}$$

and

$$y'' = \sum_{j=2}^{(i+c)-1} j(j-1)a_j x^{j-2} \tag{4}$$

Combining (2) and (3) generates the differential system

$$y'' = \sum_{j=0}^{(i+c)-1} j(j-1)a_j x^{j-2} = f(x, y, y') \tag{5}$$

Collocating (5) at $x = x_{n+i}, \; i = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$ and interpolating (2) at $x = x_{n+i}, i = 0, \frac{1}{2}$ gives a system of non-linear equation of the form

$$AX = U \tag{6}$$

where

$$A = [a_0, a_1, a_2, a_3, a_4, a_5, a_6]^T$$

$$U = \left[f_n, f_{n+\frac{1}{2}}, f_{n+1}, f_{n+\frac{3}{2}}, f_{n+2}, y_n, y_{n+\frac{1}{2}} \right]^T$$ and

$$X = \begin{bmatrix} 0 & 0 & 2 & 6x_n & 12x^2{}_n & 20x^3{}_n & 30x^4{}_n \\ 0 & 0 & 2 & 6x_{n+\frac{1}{2}} & 12x^2_{n+\frac{1}{2}} & 20x^3_{n+\frac{1}{2}} & 30x^4_{n+\frac{1}{2}} \\ 0 & 0 & 2 & 6x_{n+1} & 12x^2_{n+1} & 20x^3_{n+1} & 30x^4_{n+1} \\ 0 & 0 & 2 & 6x_{n+\frac{3}{2}} & 12x^2_{n+\frac{3}{2}} & 20x^3_{n+\frac{3}{2}} & 30x^2_{n+\frac{3}{2}} \\ 0 & 0 & 2 & 6x_{n+2} & 12x^2_{n+2} & 20x^3_{n+2} & 30x^4_{n+2} \\ 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_{n+\frac{1}{2}} & x^2_{n+\frac{1}{2}} & x^3_{n+\frac{1}{2}} & x^4_{n+\frac{1}{2}} & x^5_{n+\frac{1}{2}} & x^6_{n+\frac{1}{2}} \end{bmatrix}$$

Using Gaussian elimination method to solve for $a_j's$ in (6) gives a continuous multistep method in the form:

$$y(t) = \alpha_\theta(t) y_{n+\theta} + h\left[\sum_{\eta=0}^{\kappa} \beta(t) f_{n+\eta} + \beta_\kappa(t) f_{n+\eta} \right]$$

Where $\theta = 0, \frac{1}{2}, \eta = 0, \frac{1}{2}\left(\frac{1}{2}\right)2$, $f_{n+\kappa} = f\left(x_n + \kappa h\right)$

Then, using the transformation $t = \frac{x - x_n}{h}$ we have $\frac{dt}{dx} = \frac{1}{h}$ where $y_{n+\theta}$, $y_{n+\kappa}$, $f_{n+\eta}$ and $f_{n+\kappa}$ give a continuous scheme and the coefficients are put as follows:

$$
\begin{aligned}
\alpha_0 &= -2t - 1 = (-1 - 2t) \\
\alpha_{1/2} &= 2t + 2 \\
\beta_0 &= \frac{h^2}{2880}\left[57 + 97t + 80t^3 - 40t^4 - 96t^5 + 64t^6\right] \\
\beta_{1/2} &= \frac{h^2}{720}\left[153 + 361t - 160t^3 + 160t^4 + 48t^5 - 64t^6\right] \\
\beta_1 &= \frac{h^2}{480}\left[7 + 111t + 240t^2 - 200t^4 + 64t^6\right] \\
\beta_{3/2} &= \frac{h^2}{2880}\left[3 - 13t + 160t^3 + 160t^4 - 48t^5 + 64t^6\right] \\
\beta_2 &= \frac{h^2}{2880}\left[-3 + 5t - 80t^3 - 40t^4 + 96t^5 + 64t^6\right]
\end{aligned} \tag{7}
$$

Differentiating (7) we have:

$$
\begin{aligned}
\alpha'_0 &= -\frac{2}{h} \\
\alpha'_{1/2} &= \frac{2}{h} \\
\beta'_0 &= \frac{h}{2880}\left[97 + 240t^2 - 160t^3 - 480t^4 + 384t^5\right] \\
\beta'_{1/2} &= \frac{h}{720}\left[361 - 480t^2 + 640t^3 + 240t^4 + 384t^4\right] \\
\beta'_1 &= \frac{h}{480}\left[111 + 480t - 800t^3 + 384t^5\right] \\
\beta'_{3/2} &= \frac{h}{720}\left[-13 + 480t^2 + 640t^3 - 240t^4 + 384t^5\right] \\
\beta'_2 &= \frac{h}{2880}\left[5 - 240t^2 - 160t^3 + 480t^4 + 384t^5\right]
\end{aligned} \tag{8}
$$

Evaluating (7) and (8) at $t = 1$ which implies that $x = x_{n+2}$ gives

$$y_{n+2} - 4y_{n+1/2} + 3y_n = \frac{h^2}{480}\left[7f_{n+2} + 132f_{n+3/2} + 222f_{n+1} + 332f_{n+1/2} + 27f_n\right]. \qquad (9)$$

with order $p = 5,$ Error constant, $c_7 = \frac{108507}{1666666667}$ or 6.51×10^{-5}

The first derivative is given as:

$$y'_{n+2} = \frac{1}{h}\left(-2y_n + 2y_{n+1/2}\right) + \frac{h}{2880}\left[81f_n + 1508f_{n+1/2} + 1050f_{n+1} + 5004f_{n+3/2} - 469f_{n+2}\right] \qquad (10)$$

3 Taylor's Series Algorithm for the Implementation of the Method

To generate y values for the approximate solution, the scheme and its first derivative are expanded term by term, up to the order of the scheme, by Taylor series gives:

$$y_{n+i} = y(x_n + ih) = y_n + ihy'(x_n) + \frac{(ih)^2}{2!}f_n + \frac{(ih)^3}{3!} + \frac{(ih)^4}{4!} + \frac{(ih)^5}{5!}\cdots$$

$$y'_{n+i} = y'(x_n) + ihf_n + \frac{(ih)^2}{2!}f'_n + \cdots$$

and

$$f_{n+i} = y''(x_n + ih), y''(x_n + ih) = f_n + ihf'_n + \frac{(ih)^2}{2!}f''_n + \frac{(ih)^3}{3!}f'''_n + \frac{(ih)^4}{4!}f_n^{iv} + \frac{(ih)^5}{5!}f_n^{v} + \cdots$$

where

$$f_n = f(x_n, y_n, y'_n),\ f_n^{(i)} = f^{(i)}(x_n, y_n, y'_n), i = 1, 2, 3\cdots$$

f', f'' and f'''

by partial derivatives are:

$$f' = \frac{df}{dx} = \frac{\partial f}{\partial x} + y'_n\frac{\partial f}{\partial y} + f\frac{\partial f}{\partial y'}$$

$$f'' = \frac{d^2 f}{dx^2} = 2(Ay' + Bf) + Cfy' + D + E$$

$$f''' = \frac{d^3 f}{dx^3} = 2G + 3(Hy' + If) + Jfy' + K + L + M$$

$$f^{iv} = \frac{d^4 f}{dx^4} = N + 4fO + Pf' + Q(y')^2 + R + S + T + U + V + W$$

$$f^{v} = \frac{d^5 f}{dx^5} = X + Y + Z + a + b + c + d + 6e + g + h + i + j + k$$

where

$$A=\frac{\partial^2 f}{\partial x\partial y}+f\,\frac{\partial^2 f}{\partial y\partial y'}\,,\quad B=\frac{\partial^2 f}{\partial x\partial y'}\;C=\frac{\partial f}{\partial x}+y'\frac{\partial f}{\partial y}+f\,\frac{\partial f}{\partial y'}\;D=\frac{\partial^2 f}{\partial x^2}+\left(y'\right)^2\frac{\partial^2 f}{\partial y^2}+f^2\,\frac{\partial^2 f}{\left(\partial y'\right)^2}$$

$$E=f\,\frac{\partial y}{\partial y}\,,\qquad G=y'f'\frac{\partial^2 f}{\partial y\partial y'}+f'\frac{\partial^2 f}{\left(\partial y'\right)^2}+y'ffy'\frac{\partial^2 f}{\partial y\partial y'}+f'\frac{\partial^2 f}{\partial x\partial y}$$

$$H=\frac{\partial^3 f}{\partial x\partial y}+y'\frac{\partial^3 f}{\partial x\partial y^2}+f\,\frac{\partial^2 f}{\partial y^2}+y'f\,\frac{\partial^3 f}{\partial y^2\partial y'}+f^2\,\frac{\partial^3 f}{\partial y\left(\partial y'\right)^2}+2\frac{\partial^3 f}{\partial x\partial y\partial y'}$$

$$I=\frac{\partial^3 f}{\partial x^2\partial y'}+\frac{\partial^2 f}{\partial x\partial y}+f\,\frac{\partial^3 f}{\partial y\left(\partial y'\right)^2}+f\,\frac{\partial^2 f}{\partial y\partial y'}\,,\quad J=f\,\frac{\partial f}{\partial y}+\partial y'\frac{\partial^2 f}{\partial x\partial y}+f\,\frac{\partial^3 f}{\partial y\left(\partial y'\right)^2}+f\,\frac{\partial^2 f}{\partial y\partial y'}$$

$$K=\left(\frac{\partial f}{\partial x}+y'\frac{\partial f}{\partial y}+f\,\frac{\partial f}{\partial y'}\right)\left[\frac{\partial^2 f}{\partial x\partial y'}+y'\frac{\partial^2 f}{\partial y\partial y'}+f\,\frac{\partial^2 f}{\left(\partial y'\right)}\right]\;L=\frac{\partial^3 f}{\partial x^3}+f^3\,\frac{\partial^3 f}{\left(\partial y'\right)^3}+\left(y'\right)^3\frac{\partial^3 f}{\partial y^3}\;M=f'\frac{\partial f}{\partial y}$$

$$N=\left(y'\right)^3\,\frac{\partial^4 f}{\partial y^4}+\;f^5\,\frac{\partial^4 f}{\left(\partial y'\right)^4},\,O=4\,f\,\left[\frac{\partial^4 f}{\partial x^2\partial y}+\frac{\partial^4 f}{\partial x^3\partial y'}\right]$$

$$P=f'\left[\begin{array}{l}2\,\frac{\partial^2 f}{\left(\partial y'\right)^2}+\left(f'\right)\frac{\partial^2 f}{\left(\partial y'\right)^2}+2\left(f'\right)\frac{\partial^2 f}{\partial x\partial y}+9\,y'\\ \frac{\partial^3 f}{\partial x\partial y\partial y'}+6\,\frac{\partial^3 f}{\partial x^2\partial y'}+\\ 6\left(y'\right)^2\,\frac{\partial^3 f}{\partial y^2\partial y'}+2\,y'\,\frac{\partial^2 f}{\partial y}+\\ 2\,\frac{\partial^2 f}{\partial x\partial y}+4\,f\,\frac{\partial^2 f}{\partial y\partial y'}+\\ 4\,fy'\,\frac{\partial^3 f}{\partial x\left(\partial y'\right)^2}+2\,f\,\frac{\partial^3 f}{\partial x^2\partial y}\end{array}\right]$$

$$Q=\left(y'\right)^2\left[5\frac{\partial^3 f}{\partial y^3}+4\left(y'\right)\frac{\partial^4 f}{\partial x\partial y^3}+4\,f\left(y'\right)\frac{\partial^4 f}{\partial y^3\partial y'}+6\frac{\partial^4 f}{\partial x^2\partial y^2}\right]$$

$$R=2\,y'\left[\begin{array}{l}6\,f\,\frac{\partial^4 f}{\partial x^2\partial y\partial y'}+4\,\frac{\partial^3 f}{\partial x\partial y^2}+4\,f\,\frac{\partial^3 f}{\partial y^2\partial y'}+\\ 6\,f\left(y'\right)\frac{\partial^4 f}{\partial x\partial y^2\partial y}+3\,f\left(y'\right)\frac{\partial^4 f}{\partial y^2\left(\partial y'\right)^2}+\\ 4\,f\,\frac{\partial^3 f}{\partial y\left(\partial y'\right)^2}+6\,f^2\,\frac{\partial^4 f}{\partial x\partial y\left(\partial y'\right)^2}+\frac{\partial^3}{\partial x\partial y^2}+\\ ff'\frac{\partial^3 f}{\partial x\partial y}+\,f\left(y'\right)\frac{\partial^3 f}{\partial y^3}+4\,f^2\,\frac{\partial^3 f}{\partial y^2\partial y'}\end{array}\right]$$

$$S = 2f^2\begin{bmatrix} 5\dfrac{\partial^3 f}{\partial x\partial y\partial y'} + 2f\dfrac{\partial^4 f}{\partial x\left(\partial y'\right)^3} + 2fy'\dfrac{\partial^4 f}{\partial y\left(\partial y'\right)^3} + \\ 3\dfrac{\partial^4 f}{\partial x^2\left(\partial y'\right)^2} + \dfrac{\partial^2 f}{\partial y^2} + f'\dfrac{\partial^3 f}{\left(\partial y'\right)^3} + \\ f'\dfrac{\partial^3 f}{\partial x\partial y\partial y'} + 3f\dfrac{\partial^4 f}{\partial y(\partial y')^2} \end{bmatrix}$$

$$T = 2f\left[3\frac{\partial^2 f}{\partial y\partial y'} + \frac{\partial^2 f}{\partial y^2} + 4\frac{\partial^3 f}{\partial x\left(\partial y'\right)^2}\right]$$

$$U = f''\left[\frac{\partial f}{\partial y} + \frac{\partial^2 f}{\partial x\partial y'} + 4f'\frac{\partial^2 f}{\partial y\partial y'} + f\frac{\partial^2 f}{\left(\partial y'\right)^2} + 2f\frac{\partial^2 f}{\partial x^2\partial y}\right]$$

$$V = 4y'\frac{\partial^4 f}{\partial x^3\partial y}, W = f'''\frac{\partial f}{\partial y'}, \quad X = \left(y'\right)^4\frac{\partial^5 f}{\partial y^5} + 5\left(y'\right)^3\frac{\partial^4 f}{\partial y^4} + 3\left(y'\right)^2\frac{\partial^4 f}{\partial y^4}$$

$$Y = \begin{bmatrix} f^6\dfrac{\partial^5 f}{\left(\partial y'\right)^5} + 5f^4\dfrac{\partial^4 f}{\left(\partial y'\right)^4} + 5f^3\dfrac{\partial^4 f}{\left(\partial y'\right)^4} + \\ f^3 f'\dfrac{\partial^4 f}{\left(\partial y'\right)^4} + 10f\dfrac{\partial^3 f}{\left(\partial y'\right)^3} + 4ff'\dfrac{\partial^3 f}{\left(\partial y'\right)^3} + \\ 2f^2 f''\dfrac{\partial^3 f}{\left(\partial y'\right)^3} + f\left(y'\right)^3\dfrac{\partial^4 f}{\partial y^4} + 10y'\dfrac{\partial^3 f}{\partial y^3} + \\ 4fy'\dfrac{\partial^3 f}{\partial y^3} + 3f'\left(y'\right)^2\dfrac{\partial^3 f}{\partial y^3} + f^2 y'\dfrac{\partial^3 f}{\partial y^3} \end{bmatrix}$$

$$Z = \begin{bmatrix} 2f''\dfrac{\partial^2 f}{\left(\partial y'\right)^2} + 2ff'''\dfrac{\partial^2 f}{(\partial y')^2} + 2f'\dfrac{\partial^2 f}{\left(\partial y'\right)^2} + \\ ff''\dfrac{\partial^2 f}{\left(\partial y'\right)^2} + 3f'y'\dfrac{\partial^2 f}{\partial y^2} + 2ff'\dfrac{\partial^2 f}{\partial y^2} + \\ 2f'\dfrac{\partial^2 f}{\partial y^2} + 2f\dfrac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

$$a = \left[f^{iv}\frac{\partial f}{\partial y'} + f'''\frac{\partial f}{\partial y}\right]$$

$$b = \left\{\begin{matrix} 5\left[y'\dfrac{\partial^5 f}{\partial x^4\partial y} + f\dfrac{\partial^5 f}{\partial x^4\partial y'} + f\left(y'\right)^2\dfrac{\partial^2 f}{\partial y^3\partial y'}\right] + \\ \left(y'\right)^3\begin{bmatrix} \dfrac{\partial^5 f}{\partial x\partial y^4} + 16f\dfrac{\partial^5 f}{\partial x\partial y^3\partial y'} + y''\dfrac{\partial^5 f}{\partial y^4\partial y'} \\ +10\dfrac{\partial^5 f}{\partial x^2\partial y^3} + 6f'\dfrac{\partial^4 f}{\partial y^3\partial y'} + 4f'\dfrac{\partial^4 f}{\partial x^3\partial y'} + \\ 4f\dfrac{\partial^5 f}{\partial x\partial y^3\partial y'} + 10f^2\dfrac{\partial^5 f}{\partial y^3(\partial y')^2} \end{bmatrix} \end{matrix}\right\}$$

$$c=\left\{(y')^2\begin{bmatrix}20f\dfrac{\partial^4 f}{\partial y^3\partial y'}+10f^3\dfrac{\partial^5 f}{\partial y^2(\partial y')^3}+6\dfrac{\partial^5 f}{\partial x^3\partial y^2}+\\17\dfrac{\partial^4 f}{\partial x\partial y^3}+4\dfrac{\partial^2 f}{\partial x^3\partial y^2}+6ff'\dfrac{\partial^4 f}{\partial y(\partial y')^2}+\\20f\dfrac{\partial^4 f}{\partial y^2(\partial y')^2}+9\dfrac{\partial^4 f}{\partial x\partial y^3}\end{bmatrix}+f^5\left[\dfrac{\partial^5 f}{\partial x(\partial y')^4}+y'\dfrac{\partial^5 f}{\partial y(\partial y')}\right]\right\}$$

$$d=\left\{\begin{array}{l}f^3\begin{bmatrix}6\dfrac{\partial^5 f}{\partial x^3(\partial y')^2}+25y'\dfrac{\partial^4 f}{\partial x(\partial y')^4}+4y'\dfrac{\partial^5 f}{\partial x(\partial y')^4}+\\10\dfrac{\partial^4 f}{\partial y(\partial y')^3}+4\dfrac{\partial^5 f}{\partial x(\partial y')^4}+f'\dfrac{\partial^4 f}{\partial x\partial y(\partial y')^2}\end{bmatrix}+f^2\left[4\dfrac{\partial^5 f}{\partial x^3(\partial y')^2}+25y'\dfrac{\partial^4 f}{\partial y(\partial y')^3}+25\dfrac{\partial^4 f}{\partial x(\partial y')^3}\right]\\+8f\dfrac{\partial^4 f}{\partial x^3\partial y}+8fy'\dfrac{\partial^5 f}{\partial x^3\partial y\partial y'}+4(y')^4\left[\dfrac{\partial^5 f}{\partial x\partial y^4}+f\dfrac{\partial^5 f}{\partial y^4\partial y'}\right]+10f^2\left[\dfrac{\partial^4 f}{\partial x^2\partial y\partial y'}+y'\dfrac{\partial^2 f}{\partial x\partial y^2\partial y'}+3(y')^2\dfrac{\partial^5 f}{\partial x\partial y^2(\partial y')^2}\right]\end{array}\right\}$$

$$e=\left\{f\begin{bmatrix}y'\dfrac{\partial^4 f}{\partial x^2\partial y^2}+2y'\dfrac{\partial^5 f}{\partial x^3\partial y\partial y'}+4y'\dfrac{\partial^4 f}{\partial x\partial y^2\partial y'}+\\2\dfrac{\partial^4 f}{\partial x^3(\partial y')^2}+y'\dfrac{\partial^3 f}{\partial y^2\partial y}\end{bmatrix}+f'\begin{bmatrix}3y'\dfrac{\partial^4 f}{\partial x^2\partial y\partial y'}+\dfrac{\partial^4 f}{\partial x^3\partial y'}+f\dfrac{\partial^4 f}{\partial x^2(\partial y')^2}+\\y'\dfrac{\partial^3 f}{\partial y^2\partial y'}+2y'\dfrac{\partial^3 f}{\partial y^2\partial y'}+(y')^2\dfrac{\partial^4 f}{\partial x\partial y^2\partial y'}\end{bmatrix}\right\}$$

$$g=\left\{\begin{array}{l}6f^2\begin{bmatrix}\dfrac{\partial^5 f}{\partial x^3(\partial y')}+4y'\dfrac{\partial f}{\partial x^2\partial y(\partial y')^2}+\dfrac{\partial^5 f}{\partial x\partial y(\partial y')^2}+\\y'\dfrac{\partial^4 f}{\partial x\partial y^2\partial y'}+4\dfrac{\partial^4 f}{\partial x^2\partial y\partial y'}+\dfrac{\partial^3 f}{\partial x\partial y^2}+3\dfrac{\partial^3 f}{\partial y^2(\partial y')^2}\end{bmatrix}+(f')^2\begin{bmatrix}4\dfrac{\partial^2 f}{\partial y\partial y'}+y'\dfrac{d^3 f}{\partial y(\partial y')^2}+f\dfrac{\partial^3 f}{(\partial y')^3}+\\2\dfrac{\partial^3 f}{\partial x^2\partial y}+4y'\dfrac{\partial^3 f}{\partial x\partial y}+2\dfrac{\partial^3 f}{\partial x^2\partial y}+2f\dfrac{\partial^3 f}{\partial x\partial y\partial y'}\end{bmatrix}+\\f\begin{bmatrix}y'\dfrac{\partial^4 f}{\partial x\partial y^2\partial y'}+16(y')^2\dfrac{\partial^5 f}{\partial x^2\partial y^2\partial y'}+2\dfrac{\partial^3 f}{\partial x\partial y}+\\8\dfrac{\partial^3 f}{\partial x\partial y^2}+8y'\dfrac{\partial^4 f}{\partial x\partial y^2\partial y'}+16y'\dfrac{\partial^3 f}{\partial y^2\partial y'}\end{bmatrix}+\begin{bmatrix}5f^2(y')^2\dfrac{\partial^4 f}{\partial y^3\partial y'}+8f^2\dfrac{\partial^3 f}{\partial x^2\partial y'}+\\2(f')^2\dfrac{\partial^3 f}{\partial x(\partial y')^2}+8f^2y'\dfrac{\partial^4 f}{\partial y^2(\partial y')^2}\end{bmatrix}+6f^3y'\left[\dfrac{\partial^4 f}{\partial y^2(\partial y')^2}+2\dfrac{\partial^5 f}{\partial x\partial y(\partial y')^3}\right]\end{array}\right\}$$

$$h=\left\{\begin{array}{l}f''y'\begin{bmatrix}2\dfrac{\partial^3 f}{\partial x\partial y^2}+5f\dfrac{\partial^3 f}{\partial y(\partial y')^2}+10\dfrac{\partial^3 f}{\partial x\partial y\partial y'}\\+4\dfrac{\partial^3 f}{\partial x\partial y\partial y'}+4f'\dfrac{\partial^3 f}{\partial y^2\partial y'}+2f\dfrac{\partial^3 f}{\partial x\partial y^2}\end{bmatrix}+2fy'\begin{bmatrix}\dfrac{\partial^3 f}{\partial x\partial y}+f\dfrac{\partial^3 f}{\partial y^2\partial y'}+5\dfrac{\partial^3 f}{\partial y(\partial y')^2}+\\2f\dfrac{\partial^3 f}{\partial y^2\partial y'}+2\dfrac{\partial^3 f}{\partial x\partial y^2}+9\dfrac{\partial^4 f}{\partial x^2\partial y\partial y'}+\\13f\dfrac{\partial^4 f}{\partial x\partial y(\partial y')^2}+5f^2\dfrac{\partial^4 f}{\partial y(\partial y')^3}+f\dfrac{\partial^4 f}{\partial x\partial y(\partial y')^3}\\+2f\dfrac{\partial^4 f}{\partial x^2\partial y^2}+f^2\dfrac{\partial^4 f}{\partial x\partial y^2\partial y'}\end{bmatrix}\\+f'''\left[4f\dfrac{\partial^3 f}{\partial y(\partial y')^2}\right]+20y'\dfrac{\partial^4 f}{\partial x^2\partial y^2}+7f''\dfrac{\partial^3 f}{\partial x^2\partial y'}\end{array}\right\}$$

$$i = f'\left\{\begin{bmatrix} \frac{\partial^3 f}{\partial x(\partial y')^2} + 3\frac{\partial^2 f}{\partial y \partial y'} + \frac{\partial^3 f}{\partial y(\partial y')^2} + \\ 4f''\frac{\partial^3 f}{\partial x(\partial y')^3} + 2f^2\frac{\partial^3 f}{\partial x(\partial y')^2} + 2\frac{\partial^2 f}{\partial x \partial y} \end{bmatrix} + f\begin{bmatrix} \frac{\partial^4 f}{\partial x^2(\partial y')^2} + 19\frac{\partial^3 f}{\partial x \partial y \partial y'} + 4\frac{\partial^2 f}{\partial y \partial y'} + \\ 2\frac{\partial^4 f}{\partial x^3 \partial y} + 8\frac{\partial^3 f}{\partial x(\partial y')^2} + 4f^2\frac{\partial^4 f}{\partial x^2 \partial y \partial y'} \end{bmatrix}\right\}$$

$$j = ff''\left\{\begin{bmatrix} 6\frac{\partial^3 f}{\partial x(\partial y')^2} + 2\frac{\partial^4 f}{\partial x(\partial y')^3} + \\ f'\frac{\partial^3 f}{\partial y(\partial y')^2} + 2\frac{\partial^2 f}{\partial x \partial y} \end{bmatrix}\right\} + ff''\begin{bmatrix} 3\frac{\partial^2 f}{\partial x \partial y} + f\frac{\partial^2 f}{\partial y \partial y'} + \\ 4\frac{\partial^3 f}{\partial x^2 \partial y} + f\frac{\partial^3 f}{\partial x \partial y \partial y'} \end{bmatrix}$$

$$k = \left\{\begin{array}{l} 2f'''\left[\frac{\partial^3 f}{\partial x \partial y} + 2f'\frac{\partial^2 f}{\partial y \partial y'} + f\frac{\partial^2 f}{\partial x \partial y}\right] + 2f(y')^2\left[f'\frac{\partial^4 f}{\partial x \partial y^3} + \frac{\partial^4 f}{\partial x \partial y}\right] + \\ 4f^2\left[8\frac{\partial^3 f}{\partial y(\partial y')^2} + \frac{\partial^3 f}{\partial x \partial y^2}\right] + fy'\left[\frac{\partial^4 f}{\partial x \partial y(\partial y')^2}\right] + \begin{bmatrix} 28f^3\frac{\partial^4 f}{\partial x \partial y(\partial y')^2} + 5f^3\frac{\partial^3 f}{\partial y^2 \partial y'} + \\ 2f^2 f'\frac{\partial^3 f}{\partial x \partial y^2} + 26f\frac{\partial^3 f}{\partial x \partial y \partial y'} + 4(f'')^2\frac{\partial^2 f}{\partial y \partial y'} \end{bmatrix} \end{array}\right\}$$

4 Analysis of the Basic Properties of Method

In verifying the accuracy and applicability of our method, we examine the basic properties which include order, error constant, consistency and zero stability.

4.1 Order and error constant

Definition 1: According to Lambert [1], Linear Multistep Method (9) is said to be of order p, if p is the largest positive integer for which $c_0 = c_1 = c_2 = \cdots = c_p = c_{p+1} = 0$ but $c_{p+2} \neq 0$.

Expanding (9) by Taylor series and comparing coefficients of the expansion equating it to zero, we get the c_i values for the method:

$$c_0 = c_1 = c_2 = \cdots = c_6$$

Hence, the method is of order $p = 5$ with principal truncation error $c_{p+2} = -\frac{108507}{1666666667} \approx 6.51 \times 10^{-5}$

4.2 Consistency

For (9) to be consistent, the following criteria must be met.

Condition 1: $p \geq 1$

Condition 2: $\sum_{j=0}^{k} \alpha_j = 0$ where $j = (0 \cdots 2)$

Condition 3: $\rho'(r) = 0$ when $r = 1$

Condition 4: $\rho''(r) = 2!\sigma(r)$ when $r = 1$

where ρ and σ are the first and second characteristic polynomials of (9), applying these conditions to (9), the method was found to be consistent.

4.3 Zero stability

Definition 2 (Lambert [1]**):** A linear multistep method is said to be zero-stable if no root $\rho(r)$ has modulus greater than one (that is, if all roots of $\rho(r)$ lie in or on the unit circle). A numerical solution to class of system (1) is stable if the difference between the numerical and the theoretical solutions can be made as small as possible. Hence, (9) is found to be zero-stable since none of the roots has modulus greater than one.

4.4 Convergence

Definition 3: A linear multistep method of the form (9) is convergent if it is consistent and zero stable. Hence the necessary and sufficient conditions for the method (9) to be convergent is that it must both be consistent and zero-stable. Since, these conditions are satisfied, then the method (9) is said to be convergent.

5 Numerical Experiments

The accuracy of the method (9) for the direct solution of (1) is tested on some linear and non-linear problems.

Problem 1:

$$y'' = \frac{(y')^2}{2y} - 2y,\ y\left(\frac{\pi}{6}\right) = \frac{1}{4},\ y'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2},\ h = \frac{1}{320}$$

Analytical Solution

$$y(x) = Sin^2 x$$

Problem 2:

$$y'' = x(y')^2\quad y(0) = 1,\ y'(0) = \frac{1}{2},\ h = \frac{1}{320}$$

Analytical Solution

$$y(x) = 1 + \frac{1}{2}\log\left(\frac{2+x}{2-x}\right)$$

Problem 3:

The temperature y degrees of a body, t minutes after being placed in a certain room, satisfies the differential equation $3\frac{d^2y}{dt^2}+\frac{dy}{dt}=0$. By using the substitution $z=\frac{dy}{dt}$, or the otherwise, find y in terms of t given that $y=60$ when $t=0$ and $y=35$ when $t=6\ln 4$. Find after how many minutes the rate of cooling of the body will have fallen below one degree per minute, giving your answer correct to the nearest minute. How cool does the body get?

Formulating the Problem, we have;

$$y''=\frac{(-y')}{3},\quad y(0)=60,\ y'(0)=-\frac{80}{9},\ h=\frac{1}{320}$$

Analytical Solution

$$y(x)=\frac{80}{3}e^{-\left(\frac{1}{3}\right)x}+\frac{100}{3}$$

6 Results

The numerical solution of problems 1, 2 and 3 are presented in the tables below. The comparison of the errors in our new method with the work of existing authors are also shown here.

Table 1. Showing the numerical result for problem 1

x	Exact solution	Computed solution	Error
0.1	0.2554320488157864	0.2554320488157865	1.11E-16
0.2	0.2581624564719335	0.2581624564719332	3.88E-16
0.3	0.2609023108763734	0.2609023108763727	7.21E-16
0.4	0.2636515050038920	0.2636515050038909	1.05E-15
0.5	0.2664099314644430	0.2664099314644412	1.83E-15
0.6	0.2691774825073438	0.2691774825073411	2.72E-15
0.7	0.2719540500254835	0.2719540500254801	3.44E-15
0.8	0.2747395255595467	0.2747395255595421	4.55E-15
0.9	0.2775338003022494	0.2775338003022436	5.77E-15
1.0	0.2803367651025898	0.2803367651025824	7.32E-15

Table 2. Comparison of errors in our new method and Kayode [15] for problem 1

X	Error in new method	Error in Kayode [15]
0.1	1.110223E-16	0.64811445E-07
0.2	3.885781E-16	0.80343529E-07
0.3	7.216450E-16	0.93317005E-07
0.4	1.054712E-15	0.10334724E-06
0.5	1.831868E-15	0.11012633E-06
0.6	2.720044E-15	0.11342972E-06
0.7	3.441691E-15	0.11312237E-06
0.8	4.551914E-15	0.10916432E-06
0.9	5.773160E-15	0.10161543E-06
1.0	7.327472E-15	0.90639024E-07

Table 3. Showing the numerical result for problem 2

X	Exact solution	Computed solution	Error
0.1	1.05004172927849	1.05004172927849	0.000000E+000
0.2	1.10033534773107	1.10033534773106	6.217249E-15
0.3	1.15114043593646	1.15114043593644	2.153833E-14
0.4	1.20273255405408	1.20273255405402	5.573320E-14
0.5	1.25541281188299	1.25541281188287	1.161293E-13
0.6	1.30951960420311	1.30951960420288	2.220446E-13
0.7	1.36544375427139	1.36544375427099	3.994582E-13
0.8	1.42364893019360	1.42364893019290	6.943335E-13
0.9	1.48470027859405	1.48470027859286	1.187273E-12
1.0	1.54930614433405	1.54930614433203	2.026601E-12

Table 4. Comparison of Errors in our New Method with Kayode [15] and Adesanya [16] for Problem 2

X	Error in new method	Error in Kayode [15]	Adesanya [16]
0.1	0.000000E+000	0.61853700E-08	6.4420468E-11
0.2	6.217249E-15	0.31695117E-07	5.4567017E-10
0.3	2.153833E-14	0.75714456E-07	1.921674E-09
0.4	5.573320E-14	0.14304432E-06	4.797029E-09
0.5	1.161293E-13	0.24120724E-06	9.998000E-09
0.6	2.220446E-13	0.38177170E-06	1.871478E-08
0.7	3.994582E-13	0.58268768E-06	3.272868E-08
0.8	6.943335E-13	0.87233773E-07	5.4792477E-08
0.9	1.187273E-12	0.12968951E-07	8.929446E-08
1.0	2.026601E-12	0.19343897E-06	1.4347036E-07

Table 5. Showing the numerical result for problem 3

X	Exact solution	Computed solution	Error
0.1	59.125762679520	59.125770155947	7.476427E-06
0.2	58.280186267509	58.280215661700	2.939419E-05
0.3	57.462331147625	57.462395949280	6.480165E-05
0.4	56.671288507811	56.671401298345	1.127905E-05
0.5	55.906179330416	55.906351828020	1.724976E-04
0.6	55.166153415412	55.166396518144	2.431027E-04
0.7	54.450388435647	54.450712262611	3.238270E-04
0.8	53.758089023057	53.758502953777	4.139307E-04
0.9	53.088485884845	53.088998596888	5.127120E-04
1.0	52.440834948634	52.441454453559	6.195049E-04

7 Discussion of Results

Tables 1, 3 and 5 presented the numerical solutions in terms of the maximum errors obtained for each of the problems considered respectively. The error of the new method is compared with those of predictor-corrector and block method of Kayode [15] and Adesanya [16] respectively.

In Table 2, the new method converges faster than Kayode [15] when solving problem 1 with the same order but different approach. This makes the new method to be more efficient than previous method as displayed in global maximum errors obtained for the method in (9). Also, we compared the new method with Kayode [15] and Adesanya [16] in Table 4. These authors respectively solved problem 2 with predictor-corrector and

Block mode. With our results as displayed in Table 3, the error in our new method shows to be more efficient and converges faster. Also the new method was used to solve an engineering (cooling) problem which shows that the body become more cooler and fallen below one degree ($1^0 C$) as the step length (h) of our method is been reduced which makes the temperature of the body in the room to satisfy our differential equations and the new method developed as been demonstrated in the computed result and the error displayed in Table 3.

8 Conclusion

In this paper, a new method with the use of Taylor's series for the approximation of y variables has enabled us to compute the derivatives of the method to any possible order which allows direct solution of Initial Value Problems (IVPs) of ordinary differential equations. Using this new method with all computation with the aid of MATLAB generated codes; this enables us to compute directly the solution of second order ordinary differentials equations with initial value problems (IVPs) directly without reducing to system of first order. Based on this new approach, it is evident that the new method is considerably more efficient than other numerical methods with the same properties of consistency, zero-stability and convergence.

Competing Interests

Authors have declared that no competing interests exist.

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