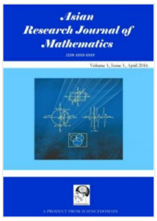

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Some Theorems on Fixed Points Set of Asymptotically Demicontractive Mappings in the Intermediate Sense

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this paper, we establish the closedness and convexity of the fixed point set of asymptotically demicontractive mappings in the intermediate sense introduced by Olaleru and Okeke [Convergence theorems on asymptotically demicontractive and hemi-contractive mappings in the intermediate sense. Fixed Point Theory and Applications 2013, 2013:352]. We also establish the equality of the fixed point set and asymptotic fixed point set.

Keywords: Convex sets; demiclosed; demicontractive mappings in the intermediate sense; fixed point set; asymptotic fixed point set.

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1 Introduction

The classes of asymptotically demicontractive maps and asymptotically hemicontractive maps were first introduced in 1987 by Liu [1], and they properly contain the class of asymptotically pseudocontractive maps and asymptotically strict pseudocontractive maps in which the fixed point set $F(T)$ is nonempty. Xiaolong et al. proved some convergence theorems on asymptotically pseudocontractive mappings in the intermediate sense [2]. Zhang gave a strong convergence criteria for the class of the mappings [3].

In 1991, Shcu introduced the class of asymptotically pseudocontractive maps [4].

In 2013, Olaleru and Okeke introduced new classes of nonlinear maps; asymptotically demicontractive mappings in the intermediate sense and asymptotically hemicontractive mappings in the intermediate sense as generalizations of the classes of asymptotically demicontractive mappings and asymptotically hemicontractive mappings, respectively [5].

Zhou [6] showed that every uniformly L-Lipschitzian and asymptotically pseudocontractive mapping which is also uniformly asymptotically regular has a fixed point. Moreover, the fixed point set is closed and convex.

In this paper we show this same properties for asymptotically demicontractive mapping in the intermediate sense which is a more general map.

2 Preliminaries

Definition 2.1. Convex Set

A subset C of $\mathbb{R}$ is said to be *convex* if for all $x, y \in C$, we have $\lambda x + (1-\lambda)y \in C$ for $0 \leq \lambda \leq 1$.

In what follows, we shall define C as a convex subset of a Hilbert space H.

Definition 2.2. Fixed point set [7]

We define the *fixed point set* $F(T)$ set of a map $T : C \to C$ as

$$F(T) = \{x \in C \mid Tx = x\}.$$

Subsequently, we shall simply write T for $T : C \to C$.

Definition 2.3. Asymptotic fixed point set

A point $p \in C$ is called an *asymptotic fixed point* of T if there exists a sequence $\{x_n\} \subset C$ which converges weakly to p and $\{x_n - Tx_n\}_{n=1}^{\infty}$ converges strongly to 0. The set $\hat{F}(T)$ of all such point p is called the *asymptotic fixed point set.*

Definition 2.4. $L-$Lipschitzian map [8, 9]

T is $L-$*Lipschitzian* if

$$\| Tx - Ty \| \leq L \| x - y \| \tag{2.1}$$

for some constant $L > 0$ for all $x, y \in C$ and it is *uniformly* $L-$*Lipschitzian [5]* if

$$\|T^n x - T^n y\| \leq L\|x - y\|$$

for some constant $L > 0$ for all $n \in \mathbb{N}$ and $x, y \in C$.

If $0 < L < 1$ in (2.1), then T is *strictly contractive* but *nonexpansive* if $L = 1$ in (2.1) [8].

Definition 2.5. Quasi-nonexpansive [8]

T is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\| \, \forall \, p \in F(T), \; x \in C$$

Definition 2.6. Firmly nonexpansive [8]

T is said to be *firmly nonexpansive* if

$$\|Tx - Ty\| \leq \langle x - y, Tx - Ty \rangle \forall \, x, y \in C.$$

Definition 2.7. Strict pseudocontractive *[5]*

T is strict pseudocontractive if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 = \|x - y\|^2 + k\| \, (x - Tx) - (y - Ty) \, \|^2 \quad \forall x, y \in C \tag{2.2}$$

Definition 2.8. Asymptotically nonexpansive [10]

T is *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ with $k_n \geq 1$ and $\lim k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{2.3}$$

for all integers $n \geq 0$ and $x, y, \in C$.

Definition 2.9. Asymptotically strict pseudocontractive [11]

T is *asymptotically strict pseudocontractive* if there exists a constant $k \in [0, 1)$ and a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\|^2 \leq k_n \|x - y\|^2 + k\| \, (I - T^n) \, x - (I - T^n) \, y\|^2 \qquad \forall \, x, y \in C. \tag{2.4}$$

Remark 2.1. If $k_n = 1$ and $T^n = T$ for all $n \in \mathbb{N}$ in (2.4), we have the class of strict pseudocontractive mappings as in (2.2) and if $k = 0$, we have (2.3).

Definition 2.10. Asymptotically strict pseudocontractive in the intermediate sense [12]

T is *Asymptotically strict pseudocontractive in the intermediate sense* if there exists a constant $k \in [0, 1)$ and a sequence $\{k_n\} \subset [1, \infty)$ with $\{k_n\} \to 1$ as $n \to \infty$ such that

$$\limsup_{n \to \infty} \sup_{x, y \in C} \left(\|T^n x - T^n y\|^2 - k_n \|x - y\|^2 - k\| \, (I - T^n) \, x - (I - T^n) \, y\|^2 \right) \leq 0. \tag{2.5}$$

Define

$$\zeta_n = \max \left\{ 0, \sup_{x, y \in C} \left(\|T^n x - T^n y\|^2 - k_n \|x - y\|^2 - k\| \, (I - T^n) \, x - (I - T^n) \, y\|^2 \right) \right\}.$$

Then it follows that $\zeta_n \to 0$ as $n \to \infty$ and (2.5) becomes

$$\|T^n x - T^n y\|^2 \leq k_n \|x - y\|^2 - k\| \, (I - T^n) \, x - (I - T^n) \, y\|^2 + \zeta_n, \tag{2.6}$$

$\forall \, n \geq 1, \; x, y \in C$.

Clearly, if $\zeta_n = 0$ for all $n \geq 1$ in equation (2.6), then we obtain (2.4).

Definition 2.11. Asymptotically demicontractive mappings [11]

T is asymptotically demicontractive if there exists a sequence $\{a_n\}$ such that $\lim_{n\to\infty} a_n = 1$ and $0 \leq k < 1$

$$\|T^n x - p\|^2 \leq a_n^2 \|x - p\|^2 + k\|x - T^n x\|^2 \quad \forall\, n \in \mathbb{N},\ x \in C,\ p \in F(T) \tag{2.7}$$

Definition 2.12. Asymptotically demicontractive in the intermediate sense [5]

T is asymptotically demicontractive in the intermediate sense if there exists a sequence $\{a_n\}$ such that $\lim_{n\to\infty} a_n = 1$ and $0 \leq k < 1$ if

$$\limsup_{n\to\infty} \sup_{(x,p)\in C\times F(T)} \left(\|T^n x - p\|^2 - a_n^2\|x - p\|^2 - k\|x - T^n x\|^2\right) \leq 0, \qquad \forall\, (x,p) \in C \times F(T). \tag{2.8}$$

Observe that on setting

$$v_n = \max\left\{0, \sup_{(x,p)\in C\times F(T)} \left(\|T^n x - p\|^2 - a_n^2\|x - p\|^2 - k\|x - T^n x\|^2\right)\right\} \tag{2.9}$$

for which $\lim_{n\to\infty} v_n = 0$, (2.8) becomes

$$\|T^n x - p\|^2 \leq a_n^2\|x - p\|^2 + k\|x - T^n x\|^2 + v_n \tag{2.10}$$

Remark 2.2. If $v_n 0 \quad \forall\, n$ then (2.10) becomes (2.7).

In order to prove our result we need the following lemma:

Lemma 2.1. *[8]Let H be a real Hilbert space and $x, y \in H$ with $0 \leq \lambda \leq 1$, then*

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\| + (1-\lambda)\|y\| - \lambda(1-\lambda)\|x - y\|$$

3 Main Results

Theorem 3.1. *Let H be a Hilbert space, C a nonempty closed and bounded convex subset of H and $T : C \to C$ be uniformly L-Lipschitzian and asymptotically demicontractive mapping in the intermediate sense. If $F(T^n)$ is nonempty, then $F(T^n)$ is closed and convex.*

Proof. Suppose $\{x_n\}$ is a sequence in $F(T^n)$ which converges to x, we shall show that $x \in F(T^n)$. Consider

$$\begin{aligned}
\|T^n x - x\|^2 &= \|T^n x - T^n x_n + T^n x_n - x\|^2, \\
&\leq \left(\|T^n x - T^n x_n\| + \|T^n x_n - x\|\right)^2, \\
&\leq \|T^n x - T^n x_n\|^2 + \|T^n x_n - x\|^2 + 2\|T^n x - T^n x_n\|\|T^n x_n - x\|.
\end{aligned}$$

Since $x_n \in F(T^n)$ and $T^n x_n = x_n$ then

$$\|T^n x - x\|^2 \leq \|T^n x - x_n\|^2 + \|x_n - x\|^2 + 2\|T^n x - T^n x_n\|\|x_n - x\|,$$

and since T is $L-$Lipschitzian then

$$\begin{aligned}
\|T^n x - x\|^2 &\leq \|T^n x - x_n\|^2 + \|x_n - x\|^2 + 2L\|x - x_n\|\|x_n - x\|, \\
&= \|T^n x - x_n\|^2 + (1 + 2L)\|x_n - x\|^2, \\
&\leq a_n^2\|x - x_n\|^2 + k\|x - T^n x\| + v_n + (1 + 2L)\|x_n - x\|^2, \\
(1-k)\|T^n x - x\|^2 &=\leq \left(1 + 2L + a_n^2\right)\|x_n - x\|^2 + v_n,
\end{aligned}$$

So that as $n \to \infty$,$x_n \to x$ and $v_n \to 0$, we have

$$\begin{aligned} (1-k)\,\|T^n x - x\|^2 &\to 0, \\ \implies T^n x = x &\implies x \in F(T^n). \end{aligned} \tag{3.1}$$

To show that $F(T^n)$ is convex, consider

$$z = \lambda p_1 + (1-\lambda)p_2 \quad p_1, p_2 \in F(T^n)$$

then,

$$z - p_1 = (1-\lambda)(p_2 - p_1) \qquad z - p_2 = -\lambda(p_2 - p_1). \tag{3.2}$$

Now,

$$\begin{aligned} \|T^n z - z\|^2 &= \|T^n z - (\lambda p_1 + (1-\lambda)p_2)\|^2, \\ &= \|\lambda(T^n z - p_1) + (1-\lambda)(T^n z - p_2)\|^2, \\ &= \lambda\|T^n z - p_1\|^2 + (1-\lambda)\|T^n z - p_2\|^2 - \lambda(1-\lambda)\|p_2 - p_1\|^2, \end{aligned} \tag{3.3}$$

but

$$\lambda\|T^n z - p_1\|^2 \leq \lambda a_n^2\|z - p_1\|^2 + \lambda k\|z - T^n z\|^2 + \lambda v_n, \tag{3.4}$$

and

$$(1-\lambda)\|T^n z - p_2\|^2 \leq (1-\lambda)a_n^2\|z - p_2\|^2 + (1-\lambda)k\|z - T^n z\|^2 + (1-\lambda)v_n, \tag{3.5}$$

therefore,

$$\lambda\|T^n z - p_1\|^2 + (1-\lambda)\|T^n z - p_2\|^2 \leq \lambda(1-\lambda)a_n^2\|p_2 - p_1\|^2 + k\|z - T^n z\|^2 + v_n. \tag{3.6}$$

So that

$$\|T^n z - z\|^2 \leq \lambda(1-\lambda)a_n^2\|p_2 - p_1\|^2 + k\|z - T^n z\|^2 + v_n - \lambda(1-\lambda)\|p_2 - p_1\|^2, \tag{3.7}$$

$$= \lambda(1-\lambda)\left(a_n^2 - 1\right)\|p_2 - p_1\|^2 + k\|z - T^n z\|^2 + v_n, \tag{3.8}$$

and

$$\|T^n z - z\|^2 \leq \frac{1}{1-k}\Big(\lambda(1-\lambda)\left(a_n^2 - 1\right)\|p_2 - p_1\|^2 + v_n\Big). \tag{3.9}$$

Since $a_n \to 1$ as $n \to \infty$then $a_n^2 - 1 \to 0$ and $v_n \to 0$ we have that

$$T^n z = z \implies z \in F(T^n)$$

□

Theorem 3.2. *Let $T : C \to C$ be asymptotically demicontractive mapping in the intermediate sense, then $I - T^n$ is demiclosed at 0.*

Proof. $I - T^n$ is demiclosed at 0 if $\{x_n\} \subset C$ such that $x_n \xrightarrow{w} p$ and $(I - T^n)x \to 0$. So, define

$$f : C \to [0, \infty) \tag{3.10}$$

as

$$f(x) = \limsup_{n\to\infty} \| x_n - x \|^2 \ \text{ for each} x \in H. \tag{3.11}$$

Clearly,

$$\begin{aligned} f(x) &= \limsup_{n\to\infty}\|x_n - x\|^2 \\ &= \limsup_{n\to\infty}\|x_n - p\|^2 + \|p - x\|^2 \\ &= f(p) + \|p - x\|^2 \ \forall\, p \in H \end{aligned} \tag{3.12}$$

So

$$\begin{aligned} f(T^n p) &= \limsup_{n\to\infty} \|x_n - T^n p\|^2 \\ &= \limsup_{n\to\infty} \|T^n p - x_n\|^2 \\ &\leq \limsup_{n\to\infty} \left(a_n^2 \|p - x_n\|^2 + k\|p - T^n p\|^2 + v_n\right) \end{aligned}$$

$$f(T^n p) \leq \limsup_{n\to\infty} a_n^2 \|p - x_n\|^2 + k \limsup_{n\to\infty} \|p - T^n p\|^2 + 0$$

Using 3.12 and knowing that $\limsup_{n\to\infty} a_n^2 \to 1$

$$f(T^n p) \leq f(p) + k \limsup_{n\to\infty} \|p - T^n p\|^2$$

Using 3.12 again

$$\begin{aligned} f(p) + \|p - T^n p\|^2 &\leq f(p) + k \limsup_{n\to\infty} \|p - T^n p\|^2 \\ (1-k)\,\|p - T^n p\|^2 &\leq 0. \end{aligned}$$

Thus

$$p = T^n p \implies p \in F(T^n)$$

□

Theorem 3.3. *Let $T : C \to C$ be L-Lipschitzian and asymptotically demicontractive mapping in the intermediate sense, then $F(T^n) = \hat{F}(T^n)$*

Proof. If $p \in F(T^n)$ then there exists $\{x_m\} \in F(T^n)$ such that $x_m \longrightarrow p$ and $x_m - T^n x_m \longrightarrow 0$ (since $x_m = T^n x_m \; \forall \; x_m \in F(T^n)$), thus $p \in \hat{F}(T^n)$.

If $p \in \hat{F}(T^n)$ then there exists $\{x_m\} \in F(T^n)$ such that $x_m \xrightarrow{w} p$ and $\{x_m - T^n x_m\} \longrightarrow 0$. Using theorem 3.2, $(I - T^n)\,p = 0$ and so, $p \in F(T^n)$.

Hence,

$$F(T^n) = \hat{F}(T^n)$$

□

Competing Interests

Authors have declared that no competing interests exist.

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