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Population Dynamics in Optimally Controlled Economic Growth Models: Case of Cobb-Douglas Production with Human Capital

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Authors' contributions

This work was carried out in collaboration between all authors. Author SOB designed the study, performed the statistical analysis, wrote the protocol and the first draft of the manuscript, as well as managed the analyses of the study and the literature searches. Authors FTO and GAO provided the direction of analyses. All authors read and approved the final manuscript.

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Abstract

In this paper, optimally controlled economic growth models with Cobb-Douglas aggregate production function are formulated to compare and contrast real per capita GDP performance as the population growth dynamics vary from purely exponential to strongly logistic. Using analytical and qualitative techniques, as well as numerical simulations, the population related parameters which induce qualitative changes in real per capita GDP over time are investigated. Consumption per effective labour and investments per effective labour in respect of (physical and human) capital are used as control variables, and (physical and human) capital per effective labour applied as state variables. Income per effective labour is used as the output variable. A time-discounted quadratic cost functional of the state and control vectors is used as the objective functional. It is found that, generally, under research and development (R & D) technological process, real per capita income grows faster and establishes higher time values to the extent that the population growth dynamics is purely exponential and far from strongly logistic. On the contrary, under any other case besides R & D, real per capita income grows faster and establishes higher

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time-values to the extent that the underpinning population growth dynamics is strongly logistic and far from being purely exponential. These results have far reaching implications in respect to the management of both underdeveloped (with exponential population growth) and developed (with logistic population growth) economies.

Keywords: Population dynamics; logistic growth; Malthusian growth; economic development; optimal control.

1 Introduction

The outcomes from previous studies in models using linear and Cobb-Douglas aggregate production functions of labour, technology and physical capital [1,2], offer some interesting pointers. These results indicate, inter alia, that the share of physical capital in the production mix, the rates of savings and or investments, have immense direct positive influence on the time-performance of real per capita income. Besides these observations, corroborated by age-long evidence in literature, empirical or otherwise [3,4,5,6,7,8,9,10,11], are the roles played by technology, especially its processes and growth dynamics, as well as the population growth dynamics. Also inherently noticeable in [1,2] are the playouts of the Malthusian[1] [12,13] (contradicted by [14]) and Boserupian[2] [15] concepts and concerns of the inter-relationships between population growth and economic advancement.

According to [1,2], the population dynamics parameter largely dictates, directly or indirectly, how most parameters impart on real per capita GDP. It is also found out that under R & D, economies with exponential population growth consistently perform better than those with logistic population growth, over time. How true is this finding generally, at least, theoretically, which tends to contradict the age-long theory, supported by empirical evidence, that economies with exponential population growth invariably perform worse than those with logistic population growth? Thus there is the need to extend the models in [1,2] to cover human capital, and to generalized r-factor production functions.

In response, this paper seeks to discuss the impact of population dynamics in optimally controlled economic growth models, using a Cobb-Douglas aggregate production function of labour, technology, human and physical capitals, extended to generalized r-factor production functions. It performs local stability and controllability analyses on the models. It also implements qualitative analyses on the models in respect of their dependence on the system parameters, especially, population related ones. It carries out numerical simulations to validate the theoretical results, and compares economic performance as the population growth dynamics vary from purely exponential to strongly logistic.

This paper is organized as follows: *Introduction*; *Theoretical Preliminaries*, outlining methods; *Main Results*, detailing models and results obtained; *Discussion*, discussing the results; and *Conclusion*.

2 Theoretical Preliminaries

2.1 Optimal control problem

Let $\boldsymbol{x}(t) \in \mathcal{R}^n$, $\boldsymbol{u}(t) \in \mathcal{R}^m$ and $\boldsymbol{y}(t) \in \mathcal{R}^r$ respectively denote the state, control and output vectors of a continuous time-varying controlled system. Then the state and output equations [16] are respectively

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[1] Malthus (1798) states that high population growth puts a lot of strain on economic performance, the presence of technology notwithstanding.
[2] Boserup (1965) states that population growth boosts technological growth, which in turn enhances economic well-being.

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{\eta}, t) \tag{2.1.1}$$

and

$$\boldsymbol{y}(t) = \boldsymbol{G}(\boldsymbol{x}(t), \boldsymbol{u}(t), t) \tag{2.1.2}$$

where $\boldsymbol{\eta}$ denotes the vector of system parameters. Let $V(\boldsymbol{x}, \boldsymbol{u})$ denote the associated running cost functional, and $\pi(\boldsymbol{x}(t_f), t_f)$, the terminal cost [16,17,18,19,20,21], such that

$$V(\boldsymbol{x}, \boldsymbol{u}) = \pi(\boldsymbol{x}(t_f), t_f) + \int_{t_0}^{t_f} L(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau), \tau) d\tau. \tag{2.1.3}$$

From the foregoing, the task reduces to determining the control set $\boldsymbol{u}$ that minimizes the cost functional

subject to: $\dot{\boldsymbol{x}}(t) = \boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{\eta}, t)$
for $\boldsymbol{x}(t_0) = \boldsymbol{x}_0 \geq \boldsymbol{0}$, $\boldsymbol{x}(t_f) = \boldsymbol{x}_{t_f} \geq \boldsymbol{0}$ and $\boldsymbol{y}(t) = \boldsymbol{G}(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{\eta}, t)$.

If $\boldsymbol{p}(t)$ is the co-state function, then the related Hamiltonian, H, and its associated equations [22] are

$$H = H(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{p}(t), t) = L(\boldsymbol{x}(t), \boldsymbol{u}(t), t) + \boldsymbol{p}^{\boldsymbol{T}}(t)\boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{\eta}, t) \tag{2.1.4}$$

$$\Longrightarrow \quad H_{\boldsymbol{x}} = L_{\boldsymbol{x}} + \boldsymbol{g}_{\boldsymbol{x}}^T \boldsymbol{p} = -\dot{\boldsymbol{p}} \tag{2.1.5}$$

$$H_{\boldsymbol{p}} = \boldsymbol{g} = \dot{\boldsymbol{x}} \tag{2.1.6}$$

$$H_{\boldsymbol{u}} = \boldsymbol{L}_{\boldsymbol{u}} + \boldsymbol{g}_{\boldsymbol{u}}^T \boldsymbol{p} = \boldsymbol{0} \tag{2.1.7}$$

$\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, $\boldsymbol{x}(t_f) = \boldsymbol{x}_{t_f}$ or $\boldsymbol{p}(t_f) = \boldsymbol{P}(t_f)\boldsymbol{x}(t_f)$, $\boldsymbol{y}(t) = \boldsymbol{G}(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{\eta}, t)$ for $\boldsymbol{P}(t) = \boldsymbol{P}^T(t) \geq 0$ is a matrix of order $n \times n$ [16, 17, 19, 23, 24]. If Equations (2.1.1) and (2.1.2) are linear, or linearized, with representation $\{\boldsymbol{A}(t), \boldsymbol{B}(t), \boldsymbol{C}(t), \boldsymbol{D}(t)\}$, $\boldsymbol{A} \in \mathcal{R}^{n \times n}$, $\boldsymbol{B} \in \mathcal{R}^{n \times m}$, $\boldsymbol{C} \in \mathcal{R}^{r \times n}$ and $\boldsymbol{D} \in \mathcal{R}^{r \times m}$, then

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t) \tag{2.1.8}$$

$$\text{and} \quad \boldsymbol{y}(t) = \boldsymbol{C}(t)\boldsymbol{x}(t) + \boldsymbol{D}(t)\boldsymbol{u}(t) \tag{2.1.9}$$

$$\Longrightarrow \quad L = \tfrac{1}{2}\left(\boldsymbol{x}^T(t)\boldsymbol{Q}(t)\boldsymbol{x}(t) + \boldsymbol{u}^T(t)\boldsymbol{R}(t)\boldsymbol{u}(t)\right)e^{-\gamma\tau} \quad \text{and} \quad \pi = \tfrac{1}{2}\boldsymbol{x}^T(t_f)\boldsymbol{P}(t_f)\boldsymbol{x}(t_f)$$

where $\boldsymbol{R}(t) = \boldsymbol{R}^T(t) > 0$ be in $\mathcal{R}^{m \times m}$, $\boldsymbol{P}(t) = \boldsymbol{P}^T(t) \geq 0$, $\boldsymbol{Q}(t) = \boldsymbol{Q}^T(t) \geq 0$, each of which belongs to $\mathcal{R}^{n \times n}$, and $0 < \gamma < 1$ is the discount rate [25,26,27].

Let $\breve{\boldsymbol{x}}(t) = e^{-\frac{\gamma}{2}t}\boldsymbol{x}(t)$, $\breve{\boldsymbol{u}}(t) = e^{-\frac{\gamma}{2}t}\boldsymbol{u}(t)$ and $\boldsymbol{E}(t) = \boldsymbol{A}(t) - \frac{\gamma}{2}\boldsymbol{I}$ in the linearized system, then Equation (2.1.5) to Equation (2.1.7) respectively gives

$$\breve{\boldsymbol{u}}(t) = -\boldsymbol{R}^{-1}(t)\boldsymbol{B}^T(t)\boldsymbol{p}(t) \tag{2.1.10}$$

$$\dot{\boldsymbol{p}}(t) = -\boldsymbol{Q}(t)\breve{\boldsymbol{x}}(t) - \boldsymbol{E}^T(t)\boldsymbol{p}(t) \tag{2.1.11}$$

$$\text{and} \quad \dot{\breve{\boldsymbol{x}}}(t) = \boldsymbol{E}(t)\breve{\boldsymbol{x}}(t) - \boldsymbol{B}(t)\boldsymbol{R}^{-1}(t)\boldsymbol{B}^T(t)\boldsymbol{p}(t). \tag{2.1.12}$$

Using $\boldsymbol{p}(t) = \boldsymbol{P}(t)\breve{\boldsymbol{x}}(t)$, we obtain the related Riccati equations [23,24] and its discretized form

$$-\dot{\boldsymbol{P}}(t) = \boldsymbol{E}^T(t)\boldsymbol{P}(t) + \boldsymbol{P}(t)\boldsymbol{E}(t) - \boldsymbol{P}(t)\boldsymbol{B}(t)\boldsymbol{R}^{-1}(t)\boldsymbol{B}^T(t)\boldsymbol{P}(t) + \boldsymbol{Q}(t) \tag{2.1.13}$$

and

$$\boldsymbol{P}_{i+j} = \boldsymbol{P}_i - (\boldsymbol{E}_i^T\boldsymbol{P}_i + \boldsymbol{P}_i\boldsymbol{E}_i - \boldsymbol{P}_i\boldsymbol{B}_i\boldsymbol{R}_i^{-1}\boldsymbol{B}_i^T\boldsymbol{P}_i + \boldsymbol{Q}_i)\cdot j \tag{2.1.14}$$

which is solved backward in time for a unique solution $\boldsymbol{P^j} = (\boldsymbol{P^j})^{\boldsymbol{T}} \geq \boldsymbol{0}$, at each step $\boldsymbol{j}$, if it exists.

2.2 Local stability analysis

From literature [16,24,28,29,30], the non-linear continuous time-varying system is locally completely stable if for all $t \geq 0$, the system's Jacobian matrix, $\boldsymbol{J}$, of $\boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{\eta}, t)$, is negative definite at the critical point. Alternatively, if at the critical point there exists a matrix $\boldsymbol{A}_s^T(t) = \boldsymbol{A}_s(t) < 0$, for

$$\boldsymbol{A}_s(t) = \frac{1}{2}(\boldsymbol{A}^T(t) + \boldsymbol{A}(t)) \tag{2.2.1}$$

then by Lyapunov's condition [16,30], the system is locally completely stable (true also in respect of the matrix $\boldsymbol{E}(t)$). Thus the system or the pair $(\boldsymbol{E}(t), \boldsymbol{B}(t))$ is stabilizable, if there exists a feedback gain matrix $\boldsymbol{K}(t) = \boldsymbol{R}^{-1}(t)\boldsymbol{B}^T(t)\boldsymbol{P}(t)$, for all $t \geq 0$, such the closed loop system

$$\dot{\breve{\boldsymbol{x}}}(t) = [\boldsymbol{E}(t) - \boldsymbol{B}(t)\boldsymbol{K}(t)]\breve{\boldsymbol{x}}(t) \tag{2.2.2}$$

is stable. Then the feedback or control law $\breve{\boldsymbol{u}}(t) = -\boldsymbol{K}(t)\breve{\boldsymbol{x}}(t)$ is admissible [16,24]. Alternative approach is to use purely numerical methods such as in [31,32,33] to directly or indirectly solve the problem.

2.3 Local controllability and observability analyses

The n-dimensional linear continuous time-varying system, with a transition matrix $\boldsymbol{\Phi}(t, \tau)$, is locally completely controllable [16,24], within $[t_0, t_f]$, if the matrix

$$\boldsymbol{G}(t_f, t_0) = \int_{t_0}^{t_f} \boldsymbol{\Phi}(t_f, \tau)\boldsymbol{B}(\tau)\overline{\boldsymbol{B}}^T(\tau)\overline{\boldsymbol{\Phi}}^T(t_f, \tau)d\tau \tag{2.3.1}$$

is either positive definite, or $|\boldsymbol{G}(t_f, t_0)| \neq 0$, or does not have zero as an eigenvalue. The system is stabilizable if it is controllable. From [16,24], the system is also completely observable if the matrix

$$\boldsymbol{F}(t_f, t_0) = \int_{t_0}^{t_f} \overline{\boldsymbol{\Phi}}^T(\tau, t_0)\overline{\boldsymbol{C}}^T(\tau)\boldsymbol{C}(\tau)\boldsymbol{\Phi}(\tau, t_0)d\tau \tag{2.3.2}$$

is either positive definite, or $|\boldsymbol{F}(t_f, t_0)| \neq 0$, or does not have zero as an eigenvalue. It is also detectable if it is observable. Even if the system is not completely observable, but by replacing the matrix $\boldsymbol{C}(t)$ with the matrix $\boldsymbol{Q}(t)$ such that the pair $(\boldsymbol{Q}(t), \boldsymbol{E}(t))$ is observable, then the system is detectable.

3 Main Results

3.1 Population (labour) growth dynamics

Assume population (labour), $L(t)$, naturally grows at $0 < n < 1$, with a carrying capacity $\frac{1}{\sigma} > 0$. Then

$$\frac{d}{dt}L(t) = n(1 - \sigma L(t))L(t) = N(L(t); n, \sigma). \tag{3.1.1}$$

Then the associated equilibria values are, $L_1 = 0$ and $L_2 = \frac{1}{\sigma} > 0$, just as in [1, 2]. Hence, we have $N'(L_1) = n > 0$, and thus, L_1 is an unstable equilibrium value. That is L_1 is a source.[3] However, $N'(L_2) = -n < 0$, implies L_2 is a stable equilibrium, a sink. For any initial value $L_1 < L_0 < L_2$, $L(t) > 0$ and $L(t) \to L_2$ as $t \to \infty$, and hence, $0 < L(t) < \frac{1}{\sigma}$, and that, for all $t \geq 0$, $L(t)$ is bounded given that $0 < n < 1$. Furthermore, for any $L_0 > L_2$, $L(t)$ decays steadily to L_2 over time. Equation (3.1.1) indicates that the dynamics of $L(t)$ tends exponential, when $\sigma = 0$, as noted in [1, 2]. Thus $L(t)$ bifurcates when $\sigma = 0$. $L(t)$ bifurcates again whenever $\sigma = 1$, since its trajectory tends constant when $\sigma = 1$, but $L(t)$ declines to zero over time when $\sigma > 1$, for all $t \geq 0$.

Suppose the start-up time $t_0 = 0$, and the initial value L_0 is standardized to unity. Then

$$L(t) = \frac{L_0 e^{nt}}{1+\sigma L_0(e^{nt}-1)} = \frac{e^{nt}}{1+\sigma(e^{nt}-1)} = \frac{e^{nt}}{(1-\sigma)+\sigma e^{nt}} \leq e^{nt} \tag{3.1.2}$$

$$\Rightarrow \qquad \frac{L'(t)}{L(t)} = \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)} = \frac{n}{1+\frac{\sigma}{1-\sigma}e^{nt}} \leq n \tag{3.1.3}$$

since $\sigma \geq 0$, $0 < n < 1$ and for all $t \geq 0$, $1 - \sigma + \sigma e^{nt} = 1 + \sigma\left(\frac{nt}{1!} + \frac{n^2t^2}{2!} + \cdots\right) \geq 1$, and $e^{nt} \geq 1$.

3.1.1 Sensitivity analysis on population dynamics

From the foregoing, it follows that, for all $t > 0$

$$\frac{\partial L}{\partial \sigma} = -\frac{(e^{nt}-1)e^{nt}}{[1+\sigma(e^{nt}-1)]^2} < 0 \tag{3.1.4}$$

$$\frac{\partial L}{\partial n} = \frac{(1-\sigma)te^{nt}}{[1+\sigma(e^{nt}-1)]^2} \begin{cases} > 0 & \text{for } 0 \leq \sigma < 1 \\ = 0 & \text{for } \sigma = 1 \\ < 0 & \text{for } \sigma > 1 \end{cases} \tag{3.1.5}$$

and

$$\frac{\partial L}{\partial t} = \frac{(1-\sigma)ne^{nt}}{[1+\sigma(e^{nt}-1)]^2} \begin{cases} > 0 & \text{for } 0 \leq \sigma < 1 \\ = 0 & \text{for } \sigma = 1. \\ < 0 & \text{for } \sigma > 1 \end{cases} \tag{3.1.6}$$

From (3.1.4), population, $L(t)$, is a decreasing function of the parameter σ. Thus as the population's growth tends logistic, the lower the growth rate becomes, and hence, the lower its numbers in relation to the one which grows exponentially, all things being equal.[4] Expressions (3.1.5) and (3.1.6) also indicate that $L(t)$ is an increasing function of the parameter n, and of time, t, whenever $0 \leq \sigma < 1$. At the same time, $L(t)$ is a decreasing function of n and t when $\sigma > 1$, and $L(t)$ becomes a constant function of n and t when $\sigma = 1$.

3.2 Technological growth dynamics

From [11,34], using the Cobb-Douglas production function, the assumption of balanced growth, for simplicity, and the substitutions in [2], then the modified residual form of technology, $A_1(t)$, becomes

$$\frac{\dot{A}_1(t)}{A_1(t)} = \delta - \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}. \tag{3.2.1}$$

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[3] *Additionally, $N'(L_1) = n < 0$, for all $n < 0$. Hence, $n = 0$ is a bifurcation value, and in contrast with the earlier discussion, $L(t) \to 0$ as $t \to \infty$ when $n < 0$, for $L_0 > 0$.*

[4] *Developing and least developed countries (e.g., Ethiopia, Nigeria, India, Ghana, etc.) generally tend to have exponentially growing human population, whereas advanced (or high income) economies (e.g., Belgium, Sweden, Germany, Norway, Japan, etc.) and higher middle income economies usually experience logistic population growth.*

However, from [35,36,2], if $0 < \phi \le 1$ is the research sector's average productivity, and $0 < \theta < 1$ is the fraction of existing technology used to produce new one(s), then the (modified) research and development (R & D) type of technology, $A_2(t)$, is

$$\frac{\dot{A}_2(t)}{A_2(t)} = \frac{\phi n(1-\sigma)}{(1-\theta)[1+\sigma(e^{nt}-1)]}. \tag{3.2.2}$$

Assuming an initial value A_0, in each case, normalized to unity, without any loss of generality, then

$$A_1(t) = A_0[1+\sigma(e^{nt}-1)]e^{(\delta-n)t} = [1+\sigma(e^{nt}-1)]e^{(\delta-n)t} \tag{3.2.3}$$

and $$A_2(t) = A_0\left[\frac{e^{nt}}{1+\sigma(e^{nt}-1)}\right]^{\frac{\phi}{1-\theta}} = \left[\frac{1}{(1-\sigma)e^{-nt}+\sigma}\right]^{\frac{\phi}{1-\theta}}. \tag{3.2.4}$$

On the other hand, assume a technological growth dynamics similar to that of labour such that it has a natural growth rate $0 < a < 1$ with a carrying capacity $\frac{1}{\xi} > 0$. Then technology, $A_3(t)$, is defined by

$$\frac{d}{dt}A_3(t) = a(1-\xi A_3(t))A_3(t) \tag{3.2.5}$$

$$\Rightarrow \qquad A_3(t) = \frac{A_0 e^{at}}{1+A_0\xi(e^{at}-1)} = \frac{1}{(1-\xi)e^{-at}+\xi}, \qquad \text{for } A_0 = 1. \tag{3.2.6}$$

3.2.1 Sensitivity analysis on technological growth dynamics

From Equation (3.2.3), and for all $t > 0$, we obtain

$$\frac{\partial A_1}{\partial t} = [\delta - n(1-\sigma) + (e^{nt}-1)\delta\sigma]e^{(\delta-n)t} > 0 \tag{3.2.7}$$

$$\frac{\partial A_1}{\partial \sigma} = (e^{nt}-1)e^{(\delta-n)t} > 0 \tag{3.2.8}$$

and $$\frac{\partial A_1}{\partial n} = -(1-\sigma)te^{(\delta-n)t} \begin{cases} < 0 & \text{for } 0 \le \sigma < 1 \\ = 0 & \text{for } \sigma = 1 \\ > 0 & \text{for } \sigma > 1 \end{cases}. \tag{3.2.9}$$

Similarly, for all $t > 0$, and the given domain of θ and ϕ, Equation (3.2.4) also gives

$$\frac{\partial A_2}{\partial \phi} \cong \frac{(1-\sigma)nt}{1-\theta}A_2(t) \begin{cases} > 0 & \text{for } 0 \le \sigma < 1 \\ = 0 & \text{for } \sigma = 1 \\ < 0 & \text{for } \sigma > 1 \end{cases} \tag{3.2.10}$$

$$\frac{\partial A_2}{\partial \theta} \cong \frac{(1-\sigma)n\phi}{(1-\theta)^2}A_2(t) \begin{cases} > 0 & \text{for } 0 \le \sigma < 1 \\ = 0 & \text{for } \sigma = 1 \\ < 0 & \text{for } \sigma > 1 \end{cases} \tag{3.2.11}$$

$$\frac{\partial A_2}{\partial n} = \frac{1-\sigma}{1-\theta}\cdot\frac{\phi t}{1+\sigma(e^{nt}-1)}A_2(t) \begin{cases} > 0 & \text{for } 0 \le \sigma < 1 \\ = 0 & \text{for } \sigma = 1 \\ < 0 & \text{for } \sigma > 1 \end{cases} \tag{3.2.12}$$

$$\frac{\partial A_2}{\partial t} = \frac{1-\sigma}{1-\theta}\cdot\frac{\phi n}{1+\sigma(e^{nt}-1)}A_2(t) \begin{cases} > 0 & \text{for } 0 \le \sigma < 1 \\ = 0 & \text{for } \sigma = 1 \\ < 0 & \text{for } \sigma > 1 \end{cases} \tag{3.2.13}$$

and $$\frac{\partial A_2}{\partial \sigma} = -\frac{\phi}{1-\theta}\cdot\frac{e^{nt}-1}{1+\sigma(e^{nt}-1)}A_2(t) < 0. \tag{3.2.14}$$

On the other hand, for all $t > 0$, Equation (3.2.6) gives

$$\frac{\partial A_3}{\partial t} = \frac{(1-\xi)a}{1+\xi(e^{at}-1)} A_3(t) \begin{cases} > 0 & \text{for } 0 \leq \xi < 1 \\ = 0 & \text{for } \xi = 1 \\ < 0 & \text{for } \xi > 1 \end{cases} \tag{3.2.15}$$

$$\frac{\partial A_3}{\partial \xi} = -\frac{(e^{at}-1)}{[1+\xi(e^{at}-1)]^2} e^{at} < 0 \tag{3.2.16}$$

and

$$\frac{\partial A_3}{\partial a} = \frac{(1-\xi)a}{[1+\xi(e^{at}-1)]^2} e^{at} \begin{cases} > 0 & \text{for } 0 \leq \xi < 1 \\ = 0 & \text{for } \xi = 1. \\ < 0 & \text{for } \xi > 1 \end{cases} \tag{3.2.17}$$

Just like the results in Section 3.1, the results here are same as seen in [2], correspondingly. For instance, for all $t > 0$, the residual technological process, $A_1(t)$, is an increasing function of time, t, and the parameter σ. Hence, over time, $A_1(t)$ is most likely to propel increasing growth in real per capita income, $y(t)$, for increasing values of σ, and vice versa. Furthermore, $A_1(t)$ is an increasing function of the parameter n whenever $\sigma > 1$, safe the stated caveat on σ. On the other hand, whereas $A_1(t)$ is a decreasing function of the parameter n when $0 \leq \sigma < 1$, it is a constant function of n when $\sigma = 1$. Again, $A_1(t)$ is an increasing function of δ, and $y(t)$ appreciates over time whenever $\delta - n > 0$.

The R & D technological process, $A_2(t)$, suggests that, for all $t > 0$, it is an increasing function of the parameters ϕ, θ, n (unlike in $A_1(t)$ discussed earlier), and time t, whenever $0 \leq \sigma < 1$. But $A_2(t)$ is a decreasing function of the aforementioned when $\sigma > 1$, and a constant function of these whenever $\sigma = 1$, for all $t > 0$. However, for all $t > 0$, $A_2(t)$ is a decreasing function of σ, and hence, increasing values of σ is possibly inimical to real per capita income performance under the R & D technological process. Theoretically, $A_2(t)$ generates its greatest growth performance in real per capita income when $\sigma = 0$.

Similarly, the logistic formulation of technology, $A_3(t)$, is an increasing function of the parameter a, and time, t, when $0 \leq \xi < 1$. But $A_3(t)$ is decreasing function of a and t, when $\xi > 1$, whilst it is a constant function of a and t, whenever $\xi = 1$. Clearly, for all $t > 0$, $A_3(t)$ is decreasing function of ξ. Thus $A_3(t)$ is most likely to generate the greatest growth in real per capita whenever $\xi = 0$ [2].

3.3 Optimal growth model of a closed economy

We consider a closed economy without the involvement of government, in which income, $Y(t)$, is either expended on consumption, $C(t)$, or investment, $I(t)$. Then from [37,38,39,40,41], we have

$$Y(t) = C(t) + I(t). \tag{3.3.1}$$

Let labour, L, human capital, H, physical capital, K, and technology, A be the factors of production. Assume a balanced growth with labour-augmenting technology [11,41,42,43,44,45]. Then

$$Y(t) = Y(K(t), H(t), A(t)L(t)). \tag{3.3.2}$$

The production function $Y: R^4 \mapsto R$, displays constant returns to scale, is twice differentiable in L, H, and K, and satisfies the monotonicity, diminishing marginal returns, and the Inada conditions, as re-stated in [1,11]. Also, $I(t)$ in Equation (3.3.1) may be decomposed, additively, into investment in physical capital, $I_K(t)$, and in human capital, $I_H(t)$. Thus in accordance with literature, as exemplified in [11], we have

$$I(t) = I_K(t) + I_H(t), \quad I_K(t) = \dot{K}(t) + \mu_K K(t) \quad \text{and} \quad I_H(t) = \dot{H}(t) + \mu_H H(t) \tag{3.3.3}$$

where $0 < \mu_K < 1$ is the depreciation rate of physical capital and $0 < \mu_H < 1$ is that of human capital [11,20,26,41,42,45]. Thus Equation (3.3.1) either becomes

$$Y(t) = C(t) + \dot{K}(t) + \mu_K K(t) + I_H(t) \text{ or } Y(t) = C(t) + \dot{H}(t) + \mu_H H(t) + I_K(t). \quad (3.3.4)$$

Using $\hat{y} = Y/AL$, $\hat{h} = H/AL$ and $\hat{k} = K/AL$, assuming a constant technological growth $0 < a < 1$, for simplicity, then the modified constraint equations, as in [1,2], are

$$\dot{\hat{k}}(t) = f(\hat{k}(t), \hat{h}(t)) - \left(a + \mu_{\hat{k}} + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\hat{k}(t) - \hat{c}(t) - \hat{l}_{\hat{h}}(t) \quad (3.3.5)$$

and $$\dot{\hat{h}}(t) = f(\hat{k}(t), \hat{h}(t)) - \left(a + \mu_{\hat{h}} + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\hat{h}(t) - \hat{c}(t) - \hat{l}_{\hat{k}}(t). \quad . \quad (3.3.6)$$

The modified control and state vectors are $\hat{\boldsymbol{u}}(t) = (\hat{c}(t) \quad \hat{l}_{\hat{k}}(t) \quad \hat{l}_{\hat{h}}(t))^T$ and $\hat{\boldsymbol{x}}(t) = (\hat{k}(t) \quad \hat{h}(t))^T$ respectively. If $f(\hat{k}(t), \hat{h}(t))$ is linear in $\hat{k}(t)$ and $\hat{h}(t)$, then Equations (3.3.5) and (3.3.6) give

$$\dot{\hat{\boldsymbol{x}}}(t) = \boldsymbol{g}(\hat{\boldsymbol{x}}(t), \hat{\boldsymbol{u}}(t), \boldsymbol{\eta}, t) = \boldsymbol{A}(t)\hat{\boldsymbol{x}}(t) + \boldsymbol{B}(t)\,\hat{\boldsymbol{u}}(t). \quad (3.3.7)$$

The associated transformed output equation, objective functional, V, are respectively

$$\hat{y}(t) = f(\hat{k}(t), \hat{h}(t)) = \boldsymbol{C}(t)\hat{\boldsymbol{x}}(t) \quad (3.3.8)$$

and $$V(\hat{\boldsymbol{x}}, \hat{\boldsymbol{u}}) = \frac{1}{2}\hat{\boldsymbol{x}}^T(t_f)\boldsymbol{P}(t_f)\hat{\boldsymbol{x}}(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\left(\hat{\boldsymbol{x}}^T(\tau)\boldsymbol{Q}(\tau)\hat{\boldsymbol{x}}(\tau) + \hat{\boldsymbol{u}}^T(\tau)\boldsymbol{R}(\tau)\hat{\boldsymbol{u}}(\tau)\right)e^{-\gamma\tau}d\tau. \quad (3.3.9)$$

3.3.1 The state equations

Assume a generalized Cobb-Douglas aggregate production function of the form

$$Y(t) = Y\big(K(t), H(t), A(t)L(t)\big) = \rho K^\alpha(t) H^\beta(t)[A(t)L(t)]^{1-\alpha-\beta} \quad (3.3.10)$$

where $0 < \alpha < 1$ and $0 < \beta < 1$ are respectively the share of physical and human capital in the production mix, and $\alpha + \beta < 1$. Moreover, $\rho \geq 1$, represents all other factors in the aggregate production function not accounted for here. Then by the usual transformation

$$\hat{y}(t) = \frac{Y(t)}{A(t)L(t)} = f(\hat{k}(t), \hat{h}(t)) = \rho\hat{k}^\alpha(t)\hat{h}^\beta(t). \quad (3.3.11)$$

Hence, Equations (3.3.5), and (3.3.6) respectively becomes

$$\dot{\hat{k}}(t) = \rho\hat{k}^\alpha(t)\hat{h}^\beta(t) - \left(a + \mu_{\hat{k}} + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\hat{k}(t) - \hat{c}(t) - \hat{l}_{\hat{h}}(t) \quad (3.3.12)$$

$$\dot{\hat{h}}(t) = \rho\hat{k}^\alpha(t)\hat{h}^\beta(t) - \left(a + \mu_{\hat{h}} + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\hat{h}(t) - \hat{c}(t) - \hat{l}_{\hat{k}}(t). \quad (3.3.13)$$

By using the idea that savings, and for that matter, investment, is a fraction of GDP, then we can define $I(t) = sY(t)$, which follows that $C(t) = (1 - s)Y(t)$, where s is the rate of savings at any time t, then by similar transformations, Equations (3.3.12) and (3.3.13) respectively become

$$\dot{\hat{k}}(t) = \rho s_{\hat{k}}\hat{k}^\alpha(t)\hat{h}^\beta(t) - \left(a + \mu_{\hat{k}} + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\hat{k}(t) \quad (3.3.14)$$

$$\dot{\hat{h}}(t) = \rho s_{\hat{h}}\hat{k}^\alpha(t)\hat{h}^\beta(t) - \left(a + \mu_{\hat{h}} + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\hat{h}(t) \quad (3.3.15)$$

where $s_{\hat{k}}$ and $s_{\hat{h}}$ are respectively the savings rate in respect of physical and human capital, each of which is a parameter. Moreover, $s = s_{\hat{k}} + s_{\hat{h}}$, the cumulative rate of savings.

3.4 Equilibrium and linearization analyses

At the equilibrium, $\dot{\hat{k}}(t) = \dot{\hat{h}}(t) = 0$. Thus, solving Equations (3.3.14) and (3.3.15), we obtain:

$$\hat{k}^*(t) = \left[\left(\frac{\rho s_{\hat{k}}}{a+\mu_{\hat{k}}+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}}\right)^{1-\beta}\left(\frac{\rho s_{\hat{h}}}{a+\mu_{\hat{h}}+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}}\right)^{\beta}\right]^{\frac{1}{1-\alpha-\beta}} \tag{3.4.1}$$

$$\hat{h}^*(t) = \left[\left(\frac{\rho s_{\hat{k}}}{a+\mu_{\hat{k}}+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}}\right)^{\alpha}\left(\frac{\rho s_{\hat{h}}}{a+\mu_{\hat{h}}+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}}\right)^{1-\alpha}\right]^{\frac{1}{1-\alpha-\beta}}. \tag{3.4.2}$$

as the equilibrium values of $\hat{k}(t)$ and $\hat{h}(t)$, besides the trivial equilibria $\hat{k}_e = 0$ and $\hat{h}_e = 0$, which from [1,2], is of little or no consequence. Using Equation (3.4.1), the equilibrium values of income, consumption, physical and human investments (each per effective labour), of interest, are respectively

$$\hat{y}^*(t) = \rho\left[\left(\frac{\rho s_{\hat{k}}}{a+\mu_{\hat{k}}+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}}\right)^{\alpha}\left(\frac{\rho s_{\hat{h}}}{a+\mu_{\hat{h}}+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}}\right)^{\beta}\right]^{\frac{1}{1-\alpha-\beta}} \tag{3.4.3}$$

$$\hat{c}^*(t) = (1 - s_{\hat{k}} - s_{\hat{h}})\hat{y}^*(t),\quad \hat{I}^*_{\hat{k}}(t) = s_{\hat{k}}\hat{y}^*(t),\ \text{ and }\ \hat{I}^*_{\hat{h}}(t) = s_{\hat{h}}\hat{y}^*(t). \tag{3.4.4}$$

Subsequently, linearizing around the neighbourhood of the equilibrium point of the system, we have

$$\begin{aligned}\hat{y}(t) &= \rho\hat{k}^{\alpha}(t)\hat{h}^{\beta}(t)\\ &\approx \rho\left[\left(\hat{k}^*\right)^{\alpha}\left(\hat{h}^*\right)^{\beta} + \alpha\left(\hat{k}^*\right)^{\alpha-1}\left(\hat{h}^*\right)^{\beta}\{\hat{k}(t)-\hat{k}^*\} + \beta\left(\hat{k}^*\right)^{\alpha}\left(\hat{h}^*\right)^{\beta-1}\{\hat{h}(t)-\hat{h}^*\}\right]\\ \Rightarrow \qquad \hat{z}(t) &\approx \frac{\alpha}{s_{\hat{k}}}\left(a+\mu_{\hat{k}}+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\hat{k}(t) + \frac{\beta}{s_{\hat{h}}}\left(a+\mu_{\hat{h}}+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\hat{h}(t)\end{aligned} \tag{3.4.5}$$

where $\hat{z}(t) = \hat{y}(t) - \omega(t)$, and $\omega(t) = (1-\alpha-\beta)\hat{y}^*(t)$. Consequently, we have[5]

$$\dot{\hat{\boldsymbol{x}}}(t) = \boldsymbol{A}(t)\hat{\boldsymbol{x}}(t) + \boldsymbol{B}(t)\hat{\boldsymbol{u}}(t) + \boldsymbol{\varsigma}(t) \tag{3.4.6}$$

$$\hat{z}(t) = \boldsymbol{C}(t)\hat{\boldsymbol{x}}(t). \tag{3.4.7}$$

3.5 Stability, controllability and observability of the linearized systems

For any initial value $\hat{k}(t_0)$ such that $\hat{k}_e < \hat{k}(t_0) < \hat{k}^*$, which is in the domain of interest, $\hat{k}(t) > 0$ for all $t \geq 0$, and that $\hat{k}(t) \to \hat{k}^*$ as $t \to \infty$. Thus $\hat{k}(t)$ is bounded. Also, for any initial value $\hat{k}(t_0) > \hat{k}^*$, $\hat{k}(t)$ decays down to $\hat{k}^*$. Hence, $\hat{k}(t) = \hat{k}^*$ is a sink. This is similarly true in respect of $\hat{h}(t)$.

Furthermore, for $0 < \alpha < 1$, $0 < \beta < 1$ and $\alpha + \beta < 1$, the system's Jacobian, $\boldsymbol{J}$, is given by

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[5] *For* $\boldsymbol{A}(t) = \begin{pmatrix}\left(\frac{\alpha}{s_{\hat{k}}}-1\right)\left(a+\mu_{\hat{k}}+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right) & \frac{\beta}{s_{\hat{h}}}\left(a+\mu_{\hat{h}}+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\\ \frac{\alpha}{s_{\hat{k}}}\left(a+\mu_{\hat{k}}+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right) & \left(\frac{\beta}{s_{\hat{h}}}-1\right)\left(a+\mu_{\hat{h}}+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\end{pmatrix}$, $\boldsymbol{B}(t) = \begin{pmatrix}-1 & 0 & -1\\ -1 & -1 & 0\end{pmatrix}$, $\boldsymbol{\varsigma}(t) = (1 \quad 1)^T\omega(t)$ *and* $\boldsymbol{C}(t) = \left(\frac{\alpha}{s_{\hat{k}}}\left(a+\mu_{\hat{k}}+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right) \quad \frac{\beta}{s_{\hat{h}}}\left(a+\mu_{\hat{h}}+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\right)$.

$$\boldsymbol{J}\left(\hat{k}^*, \hat{h}^*\right) = \left(a + \mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\begin{pmatrix} -(1-\alpha) & \beta\frac{s_{\hat{k}}}{s_{\hat{h}}} \\ \alpha\frac{s_{\hat{h}}}{s_{\hat{k}}} & -(1-\beta) \end{pmatrix} \tag{3.5.1}$$

$$\Rightarrow \qquad |\boldsymbol{J}| = (1-\alpha-\beta)\left(a + \mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)^2 > 0. \tag{3.5.2}$$

But given that $\boldsymbol{J}_1 = m_{\hat{k}}\left(\hat{k}^*, \hat{h}^*\right) < 0$ and $|\boldsymbol{J}| > 0$ suggest that $\boldsymbol{J}$ is negative definite, the system is locally stable equilibrium point $\left(\hat{k}^*, \hat{h}^*\right)$. This is also corroborated by the nature of eigenvalues of $\boldsymbol{J}$, which are: $-(1-\alpha-\beta)\left(a + \mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)$ and $-\left(a + \mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)$, all of which are negative. But $\left(\hat{k}^*, \hat{h}^*\right)$ is unstable equilibrium point any time $\sigma > \frac{a+n+\mu}{a+n+\mu-(a+\mu)e^{nt}}$.

Let the matrices $\boldsymbol{B}$, and $\boldsymbol{C}$ be as given earlier. Then Equations (2.2.3) and (2.2.5) respectively become[6]

$$\boldsymbol{G}\left(t_f, 0\right) = \left(\frac{s_{\hat{k}} \cdot s_{\hat{h}}}{\beta s_{\hat{k}} + \alpha s_{\hat{h}}}\right)^2 \begin{pmatrix} \frac{\beta}{s_{\hat{h}}} & 1 \\ -\frac{\alpha}{s_{\hat{k}}} & 1 \end{pmatrix} \int_0^{T_f} \begin{pmatrix} f_1(t_f, \tau) & f_2(t_f, \tau) \\ f_2(t_f, \tau) & f_3(t_f, \tau) \end{pmatrix} d\tau \begin{pmatrix} \frac{\beta}{s_{\hat{h}}} & -\frac{\alpha}{s_{\hat{k}}} \\ 1 & 1 \end{pmatrix} \tag{3.5.3}$$

$$\boldsymbol{F}\left(t_f, 0\right) = \left(\frac{s_{\hat{k}} \cdot s_{\hat{h}}}{\beta s_{\hat{k}} + \alpha s_{\hat{h}}}\right)^2 \begin{pmatrix} 1 & \frac{\alpha}{s_{\hat{k}}} \\ -1 & \frac{\beta}{s_{\hat{h}}} \end{pmatrix} \int_0^{t_f} \begin{pmatrix} 0 & 0 \\ 0 & f_4(\tau, 0) \end{pmatrix} d\tau \begin{pmatrix} 1 & -1 \\ \frac{\alpha}{s_{\hat{k}}} & \frac{\beta}{s_{\hat{h}}} \end{pmatrix}. \tag{3.5.4}$$

From Equation (3.5.3), $\left|\boldsymbol{G}\left(t_f, 0\right)\right| \neq 0$, given that the integral and matrices outside it have non-zero determinants. Thus the linearized system is locally completely controllable, and hence, stabilizable. Thus system projections are reachable and feasible. Obviously, $\left|\boldsymbol{F}\left(t_f, 0\right)\right| = 0$. But by replacing $\boldsymbol{C}(t)$ with $\boldsymbol{Q}(t)$ in Equation (2.3.2) gives $\left|\boldsymbol{F}\left(t_f, 0\right)\right| \neq 0$. Hence, from Section 2.3, the linearized system is detectable. Consequently, for all $t \geq 0$, there exists a unique solution $\boldsymbol{P}^j = (\boldsymbol{P}^j)^T \geq 0$, at each step j, to Equation (2.1.13) or (2.1.14), and hence, the existence of unique solution to the control problem.

3.6 Sensitivity and bifurcation analyses of the systems

Suppose the equilibrium trajectory of income per labour is $y^*(t)$, then

$$y^*(t) = \hat{y}^*(t)A(t) = \rho_0\left[\left(\frac{\rho s_{\hat{k}}}{a+\mu+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}}\right)^{\alpha}\left(\frac{\rho s_{\hat{h}}}{a+\mu+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}}\right)^{\beta}\right]^{\frac{1}{1-\alpha-\beta}} e^{at}. \tag{3.6.1}$$

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[6] *For* $f_1(t_f, \tau) = 2\left[\frac{1+\sigma(e^{n\tau}-1)}{1+\sigma\left(e^{nT_f}-1\right)}\right]^{-2} e^{2(a+\mu+n)(\tau-t_f)}$, $f_2(t_f, \tau) = \left(\frac{\alpha}{s_{\hat{k}}} - \frac{\beta}{s_{\hat{h}}}\right)\left[\frac{1+\sigma(e^{n\tau}-1)}{1+\sigma\left(e^{nT_f}-1\right)}\right]^{\left(\frac{\alpha}{s_{\hat{k}}}+\frac{\beta}{s_{\hat{h}}}-2\right)} e^{\left(2-\frac{\alpha}{s_{\hat{k}}}-\frac{\beta}{s_{\hat{h}}}\right)(a+\mu+n)(\tau-t_f)}$,
$f_3(t_f, \tau) = 2\left[\left(\frac{\alpha}{s_{\hat{k}}} - \frac{\beta}{s_{\hat{h}}}\right)^2 + \frac{3\alpha\beta}{s_{\hat{k}} s_{\hat{h}}}\right]\left[\frac{1+\sigma(e^{n\tau}-1)}{1+\sigma\left(e^{nt_f}-1\right)}\right]^{2\left(\frac{\alpha}{s_{\hat{k}}}+\frac{\beta}{s_{\hat{h}}}-1\right)} e^{2\left(1-\frac{\alpha}{s_{\hat{k}}}-\frac{\beta}{s_{\hat{h}}}\right)(a+\mu+n)(\tau-t_f)}$ *and*
$f_4(\tau, 0) = \left(\frac{\alpha}{s_{\hat{k}}} + \frac{\beta}{s_{\hat{h}}}\right)^2 \left(a + \mu + \frac{n(1-\sigma)}{1+\sigma(e^{n\tau}-1)}\right)^2 \left[1 + \sigma(e^{n\tau} - 1)\right]^{2\left(1-\frac{\alpha}{s_{\hat{k}}}-\frac{\beta}{s_{\hat{h}}}\right)} e^{2\left(\frac{\alpha}{s_{\hat{k}}}+\frac{\beta}{s_{\hat{h}}}-1\right)(a+\mu+n)\tau}$, *using* $t_0 = 0$.

and $$\dot{y}^*(t) = \left[a + \frac{\alpha+\beta}{1-\alpha-\beta} \cdot \frac{n^2(1-\sigma)\sigma}{(a+\mu)[1+\sigma(e^{nt}-1)]+n(1-\sigma)} \cdot \frac{e^{nt}}{1+\sigma(e^{nt}-1)}\right] y^*(t) \begin{cases} > a, & 0 < \sigma < 1 \\ = a, & \text{for } \sigma = 0{,}1 \\ < a, & \text{for } \sigma > 1 \end{cases} \tag{3.6.2}$$

$$= m(y^*(t), t; a, \alpha, \beta, \mu, n, \sigma) \tag{3.6.3}$$

for $\rho_0 = \rho A_0$. Hence, the time trajectory of y (and hence, k, and h) bifurcates when $a = 0$, or $n = 0$, or $\sigma = 0$, or $\sigma = 1$. Consequently, the time-values of y will stay constant when $\sigma = 0, 1$ and $a = 0$; nose-dive whenever $a < 0$, for all $\sigma \geq 0$; and grows up when $a > 0$, for all $\sigma \geq 0$. Whenever $a = 0$, then the time trajectory of y may rise, probably slightly, when $0 < \sigma < 1$; but may initially rise, then stay constant (when $\sigma = 0$), or fall when $\sigma > 1$. Moreover, if we put $m_1 = m/y^*$, then for all $t \geq 0$

$$\frac{\partial m_1}{\partial \beta} = \frac{\sigma}{(1-\alpha-\beta)^2} \cdot \frac{n^2(1-\sigma)e^{nt}}{(a+\mu)[1+\sigma(e^{nt}-1)]^2+n(1-\sigma)[1+\sigma(e^{nt}-1)]} = \frac{\partial m_1}{\partial \alpha} \begin{cases} > 0, & \text{for } 0 < \sigma < 1 \\ = 0, & \text{for } \sigma = 0, 1. \\ < 0, & \text{for } \sigma > 1 \end{cases} \tag{3.6.4}$$

For all $t \geq 0$, we also have

$$\frac{\partial y^*}{\partial a} = \left[t - \frac{\alpha+\beta}{1-\alpha-\beta} \cdot \frac{1+\sigma(e^{nt}-1)}{n(1-\sigma)+(a+\mu)[1+\sigma(e^{nt}-1)]}\right] y^* > 0 \tag{3.6.5}$$

$$\frac{\partial y^*}{\partial \sigma} = \frac{\alpha+\beta}{1-\alpha-\beta} \cdot \frac{ne^{nt}}{(a+\mu)[1+\sigma(e^{nt}-1)]+n(1-\sigma)} \cdot \frac{y^*}{[1+\sigma(e^{nt}-1)]} > 0 \tag{3.6.6}$$

$$\frac{\partial y^*}{\partial s_{\hat{k}}} = \frac{1}{s_{\hat{k}}}\left(\frac{\alpha}{1-\alpha-\beta}\right) y^* > 0 \quad \text{and} \quad \frac{\partial y^*}{\partial s_{\hat{h}}} = \frac{1}{s_{\hat{h}}}\left(\frac{\beta}{1-\alpha-\beta}\right) y^* > 0 \tag{3.6.7}$$

$$\frac{\partial y^*}{\partial \rho} = \frac{1}{\rho}\left(\frac{1}{1-\alpha-\beta}\right) y^* > 0 \quad \text{and} \quad \frac{\partial y^*}{\partial \hat{k}_0} = \frac{\partial y^*}{\partial \hat{h}_0} = \frac{\partial y^*}{\partial \gamma} = 0 \tag{3.6.8}$$

$$\frac{\partial y^*}{\partial n} = \frac{\alpha+\beta}{1-\alpha-\beta} \cdot \frac{(1-\sigma)[\{1+(nt-1)e^{nt}\}\sigma-1]}{[1+\sigma(e^{nt}-1)][(a+\mu)[1+\sigma(e^{nt}-1)]+(1-\sigma)n]} y^* \begin{cases} > 0, & \text{for } 0 < \sigma < 1 \\ = 0, & \text{for } \sigma = 1 \\ < 0, & \text{for } \sigma = 0 \text{ or } \sigma > 1 \end{cases} \tag{3.6.9}$$

and $$\frac{\partial y^*}{\partial \mu} = -\frac{\alpha+\beta}{1-\alpha-\beta} \cdot \frac{1+\sigma(e^{nt}-1)}{(a+\mu)[1+\sigma(e^{nt}-1)]+(1-\sigma)n} y^* < 0. \tag{3.6.10}$$

From (3.6.2) and Equation (3.6.5), we can conveniently conclude that the higher the value of a, the faster y^* grows, and vice versa. Equations (3.6.2) and (3.6.6) also suggest that for $0 < \sigma < 1$, higher values of α, as well as β, ignite faster growth in y^*. The converse is also true. However, whenever $\sigma = 0$ or $\sigma = 1$, the growth effect of β (and or α) on y^* is kept at zero, and hence, y^* remains constant, except if there is a positive technological progress, as seen in Equation (3.6.2). When $\sigma > 1$, then higher values of β or α are not incentive for growth in y^*. In general, higher values of σ generate greater values of y^* over time, except that $\sigma > 1$ is a recipe for population extinction, and hence, undesirable.

From (3.6.7), $s_{\hat{h}}$, and hence, $s_{\hat{k}}$, has a direct positive effect on y^*, and the higher it is, the higher the time-values of y^*. The division by $s_{\hat{h}}$, and thus $s_{\hat{k}}$, suggests that as $s_{\hat{h}}$ or $s_{\hat{k}}$ becomes higher and higher, the growth potential in y^*, ignited by each of these, diminishes. From the system equation, that is the linearized one given in Equations (3.4.6) to (3.4.8), the time trajectories of $\hat{k}$ and $\hat{h}$, and hence, y^*, may experience the fastest growth, before the equilibrium, whenever $\alpha > s_{\hat{k}}$ and or $\beta > s_{\hat{h}}$.

From (3.6.9), a higher value of n will be instrumental for the establishment of higher time-values of y^* if and only if $0 < \sigma < 1$. However, higher values of n are inimical to the generation of higher values of y^* when $\sigma = 0$, or $\sigma > 1$. The effect of n on the time values of y^* is neutral when $\sigma = 1$. (On the other hand, higher values of μ, affect the time-values of y negatively, as per (3.6.10), probably mainly due to its effect of establishing lower equilibrium value of y. The converse is equally true).

In the more generalized form, where the factors of $Y(t)$ are $K_l(t)$, $A(t)$ and $L(t)$, for $l = 1, 2, \dots, p$, and for each $K_l(t)$ is associated factor share β_l, rate of savings s_l, and depreciation rate μ_l, then

$$y^*(t) = \rho_0 \left[\left(\frac{\rho s_1}{a+\mu_1+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}} \right)^{\beta_1} \left(\frac{\rho s_2}{a+\mu_2+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}} \right)^{\beta_2} \cdots \left(\frac{\rho s_p}{a+\mu_p+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}} \right)^{\beta_p} \right]^{\frac{1}{1-\sum_{i=1}^{p}\beta_i}} e^{at}. \quad (3.6.11)$$

Interestingly, the sensitivity analyses here are similarly the same as done before. For instance,

$$\frac{\partial y^*}{\partial s_l} = \frac{1}{s_l}\left[\frac{\beta_l}{1-\sum_{i=1}^{p}\beta_i}\right] y^* > 0, \frac{\partial y^*}{\partial \mu_l} = -\frac{\sum_{i=1}^{p}\beta_i}{1-\sum_{i=1}^{p}\beta_i} \cdot \frac{1+\sigma(e^{nt}-1)}{(a+\mu_l)[1+\sigma(e^{nt}-1)]+(1-\sigma)n} y^* < 0 \quad (3.6.12)$$

and $$\frac{\partial m}{\partial \beta_l} = \frac{\sigma}{\left(1-\sum_{i=1}^{p}\beta_i\right)^2} \cdot \frac{n^2(1-\sigma)e^{nt}}{(a+\mu_l)[1+\sigma(e^{nt}-1)]^2+n(1-\sigma)[1+\sigma(e^{nt}-1)]} \begin{cases} > 0, & \text{for } 0 < \sigma < 1 \\ = 0, & \text{for } \sigma = 0, 1 \\ < 0, & \text{for } \sigma > 1 \end{cases} \quad (3.6.13)$$

and so on and so forth, for $l = 1, 2, \dots, p$, and for all $t \geq 0$, where $\dot{y}^*(t)/y^*(t) = m$.

3.7 Growth and convergence analyses

From our systems,

$$y(t) = A(t)\hat{y}(t) = A(t)f(\hat{k}(t), \hat{h}(t)) \quad (3.7.1)$$

$$\Rightarrow \quad \frac{\dot{y}(t)}{y(t)} = \frac{\dot{A}(t)}{A(t)} + \frac{f_{\hat{k}}(\hat{k}(t),\hat{h}(t))\hat{k}(t)}{f(\hat{k}(t),\hat{h}(t))} \cdot \frac{\dot{\hat{k}}(t)}{\hat{k}(t)} + \frac{f_{\hat{h}}(\hat{k}(t),\hat{h}(t))\hat{h}(t)}{f(\hat{k}(t),\hat{h}(t))} \cdot \frac{\dot{\hat{h}}(t)}{\hat{h}(t)}$$

i.e., $$\frac{\dot{y}(t)}{y(t)} = a + \varepsilon_{\hat{k}}(\hat{k}(t))\frac{\dot{\hat{k}}(t)}{\hat{k}(t)} + \varepsilon_{\hat{h}}(\hat{h}(t))\frac{\dot{\hat{h}}(t)}{\hat{h}(t)} \quad (3.7.2)$$

$\alpha = \varepsilon_{\hat{k}}(\hat{k}(t)) = f_{\hat{k}}\hat{k}(t)/f \in (0,1)$ and $\beta = \varepsilon_{\hat{h}}(\hat{h}(t)) \in (0,1)$ are respectively the elasticity of f in respect of physical and human capitals, measuring the share of physical and human capital in the production mix. Using (3.3.18) and (3.3.20), and as in [2], then

$$\frac{\dot{y}(t)}{y(t)} \approx a - \left(1 - \varepsilon_{\hat{k}}(\hat{k}^*) - \varepsilon_{\hat{h}}(\hat{h}^*)\right)\left(a + \mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)[\ln y(t) - \ln y^*(t)] \quad (3.7.3)$$

$$\Rightarrow \quad \frac{\dot{y}(t)}{y(t)} \approx a - (1 - \alpha - \beta)\left(a + \mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)[\ln y(t) - \ln y^*(t)]. \quad (3.7.4)$$

Generalising, as in the latter part of the previous section, with $\mu = \mu_1 = \cdots = \mu_p$, for simplicity, gives

$$\frac{\dot{y}(t)}{y(t)} \approx a - \left(1 - \sum_{i=1}^{p}\beta_i\right)\left(a + \mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)[\ln y(t) - \ln y^*(t)] \quad . \quad (3.7.5)$$

Since $1 - \alpha - \beta > 0$ and $\left(a + \mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right) > 0$, the product $(1 - \alpha - \beta)\left(a + \mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right) > 0$.[7] Subsequently, when $y(t) > y^*(t)$ real per capita GDP will grow at a rate less than the technological growth, a, however marginal. Consequently, real GDP per capita GDP takes a nose-dive whenever we have $a <$

[7] *The product on the left of the relational sign is termed the rate of convergence. See [11].*

$(1-\alpha-\beta)\left(a+\mu+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)[\ln y(t)-\ln y^*(t)]$ until $y(t)=y^*(t)$. Similarly, in the absence of technology, real per capita GDP will grow at a negative rate, of magnitude given by the product of the convergence rate and the term $\ln y(t)-\ln y^*(t)$, anytime $y(t)>y^*(t)$.

At equilibrium when $y(t)=y^*(t)$, the rate of growth of real per capita GDP is a, the technological growth rate. In the absence of technology, then real per capita GDP stagnates over time whenever $y(t)=y^*(t)$, except probably, where labour growth is logistic. Furthermore, the growth rate in the economy will exceed a anytime $y(t)<y^*(t)$ until $y(t)=y^*(t)$ is attained. Thereafter real GDP per capita grows at the rate of growth of technology, and if there exists no technology in the economy then real per capita GDP stagnates over time thereafter in the case where labour growth is exponential.

3.8 The Hamilton-Pontryagin equations of the systems

Putting $\widetilde{\boldsymbol{x}}(t)=e^{-\frac{\gamma}{2}t}\widehat{\boldsymbol{x}}(t)$, $\widetilde{\boldsymbol{u}}(t)=e^{-\frac{\gamma}{2}t}\widehat{\boldsymbol{u}}(t)$, $\tilde{\boldsymbol{\varsigma}}(t)=e^{-\frac{\gamma}{2}t}\boldsymbol{\varsigma}(t)$ Equations (2.1.10) to (2.1.12), give

$$\widetilde{\boldsymbol{u}}(t)=-\boldsymbol{R}^{-1}(t)\boldsymbol{B}^T(t)\boldsymbol{p}(t) \tag{3.8.1}$$

$$\dot{\boldsymbol{p}}(t)=-\boldsymbol{Q}(t)\widetilde{\boldsymbol{x}}(t)-\boldsymbol{E}^T(t)\boldsymbol{p}(t) \tag{3.8.2}$$

$$\begin{aligned}\dot{\widetilde{\boldsymbol{x}}}(t)&=\boldsymbol{E}(t)\widetilde{\boldsymbol{x}}(t)+\boldsymbol{B}(t)\widetilde{\boldsymbol{u}}(t)+\tilde{\boldsymbol{\varsigma}}(t)\\&=\boldsymbol{E}(t)\widetilde{\boldsymbol{x}}(t)-\boldsymbol{B}(t)\boldsymbol{R}^{-1}(t)\boldsymbol{B}^T\boldsymbol{p}(t)+\tilde{\boldsymbol{\varsigma}}(t)\end{aligned} \tag{3.8.3}$$

for $$\boldsymbol{p}(t_f)=\boldsymbol{p}_{t_f}=\widetilde{\boldsymbol{P}}(t_f)\widetilde{\boldsymbol{x}}_{t_f},\ \widetilde{\boldsymbol{x}}(t_0)=\widetilde{\boldsymbol{x}}_{t_0}\geq 0 \text{ and } \widetilde{\boldsymbol{x}}(t_f)=\widetilde{\boldsymbol{x}}_{t_f}\geq 0 \tag{3.8.4}$$

and $$\hat{z}(t)=\boldsymbol{C}(t)\widehat{\boldsymbol{x}}(t)=\boldsymbol{C}(t)\widetilde{\boldsymbol{x}}(t)e^{\frac{\gamma}{2}t}. \tag{3.8.5}$$

3.9 Solution of the Hamiltonian equations

Using the applicable value of $\boldsymbol{P}^j$, and $\boldsymbol{\zeta}^j(t)=\int_{t_0}^{t}e^{\boldsymbol{M}^j(t-\tau)}\tilde{\boldsymbol{\varsigma}}(\tau)d\tau$, then

$$\dot{\widetilde{\boldsymbol{x}}}^j(t)=\boldsymbol{M}^j\widetilde{\boldsymbol{x}}^j(t)+\tilde{\boldsymbol{\varsigma}}(t) \tag{3.9.1}$$

$\Rightarrow$ $$\widetilde{\boldsymbol{x}}^j(t)=e^{\boldsymbol{M}^j(t-t_0)}\widetilde{\boldsymbol{x}}_0+\boldsymbol{\zeta}^j(t) \tag{3.9.2}$$

$$\boldsymbol{p}^j(t)=\boldsymbol{P}^j\left[e^{\boldsymbol{M}^j(t-t_0)}\widetilde{\boldsymbol{x}}_0+\boldsymbol{\zeta}^j(t)\right] \tag{3.9.3}$$

and $$\widetilde{\boldsymbol{u}}^j(t)=-\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}^j\left[e^{\boldsymbol{M}^j(t-t_0)}\widetilde{\boldsymbol{x}}_0+\boldsymbol{\zeta}^j(t)\right]. \tag{3.9.4}$$

We obtain $\widehat{\boldsymbol{x}}^j(t)$ and $\widehat{\boldsymbol{u}}^j(t)$, and hence, $\boldsymbol{x}^j(t)$ and $\boldsymbol{u}^j(t)$, via the substitutions made, and finally recover

$$y^j(t)=\left(\omega(t)+\boldsymbol{C}(t)\left[e^{\boldsymbol{M}^j(t-t_0)}\widetilde{\boldsymbol{x}}_0+\boldsymbol{\zeta}^j(t)\right]e^{\frac{\gamma}{2}t}\right)e^{at}. \tag{3.9.5}$$

Whenever equilibrium is reached, the trajectory of y assumes the form defined in Equation (3.6.1) thereafter. The trajectories of h, k, and the control variables, follow similar traits. (The originally linear system results in a non-stable equilibria, and hence, their trajectories are unreliable, though with similar structure.) CES production functions yield models similar in structure to those seen above and below.

Assuming the technological processes defined by $A_i(t)$ above, $i = 1, 2, 3$, we obtain analogous trajectory of $y(t)$ (and similar ones for $\boldsymbol{u}(t)$, $\boldsymbol{x}(t)$) thus[8]

$$y_i^j(t) = \left(\omega_i(t) + \boldsymbol{C}_i(t)\left[e^{\boldsymbol{M}_i^j(t-t_0)}\widetilde{\boldsymbol{x}}_0 + \boldsymbol{\zeta}_i^j(t)\right]e^{\frac{\gamma}{2}t}\right)A_i(t). \tag{3.9.6}$$

In each case, the equilibrium trajectory of $y_i(t)$, for $i = 1, 2, 3$, is given by

$$y_i^{*j}(t) = \frac{1}{1-\alpha-\beta}\omega_i(t)A_i(t). \tag{3.9.7}$$

These results also apply in respect of the generalized r-factor aggregate production function used earlier.

4 Discussion

4.1 Systems with constant technological growth

Per the model results, the equilibrium value of real per capita GDP, $y(t)$, is increasingly higher, the more logistic the population growth becomes, all other things being equal. Hence, the performance of an economy becomes much better as the population growth dynamics varies through purely exponential to strongly logistic. All the simulation plots, especially Fig. 4.1 and Fig. 4.2 below demonstrate and confirm this. The difference between these two scenarios becomes markedly great over time for a blend of higher values of a, α, β, $s_{\hat{k}}$, $s_{\hat{h}}$ and σ (and ρ), and lower or declining values of n,[9] whenever $\sigma = 0$ (and μ).

[8] *For* $\omega_1(t) = \rho(1-\alpha-\beta)\left[\left(\frac{\rho s_{\hat{k}}}{\mu+\delta}\right)^{\alpha}\left(\frac{\rho s_{\hat{h}}}{\mu+\delta}\right)^{\beta}\right]^{\frac{1}{1-\alpha-\beta}}$, $\omega_2(t) = \rho(1-\alpha-\beta)\left[\left(\frac{\rho s_{\hat{k}}}{\mu+\frac{1+\phi-\theta}{1-\theta}\cdot\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}}\right)^{\alpha}\left(\frac{\rho s_{\hat{h}}}{\mu+\frac{1+\phi-\theta}{1-\theta}\cdot\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}}\right)^{\beta}\right]^{\frac{1}{1-\alpha-\beta}}$,

$\omega_3(t) = \rho(1-\alpha-\beta)\left[\left(\frac{\rho s_{\hat{k}}}{\mu+\frac{a(1-\xi)}{1+\xi(e^{at}-1)}+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}}\right)^{\alpha}\left(\frac{\rho s_{\hat{h}}}{\mu+\frac{a(1-\xi)}{1+\xi(e^{at}-1)}+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}}\right)^{\beta}\right]^{\frac{1}{1-\alpha-\beta}}$, $\boldsymbol{C}_1(t) = (\mu+\delta)\begin{pmatrix}\frac{\alpha}{s_{\hat{k}}} & \frac{\beta}{s_{\hat{h}}}\end{pmatrix}$

$\boldsymbol{C}_2(t) = \left(\mu + \frac{1+\phi-\theta}{1-\theta}\cdot\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\begin{pmatrix}\frac{\alpha}{s_{\hat{k}}} & \frac{\beta}{s_{\hat{h}}}\end{pmatrix}$, $\boldsymbol{C}_3(t) = \left(\mu + \frac{a(1-\xi)}{1+\xi(e^{at}-1)} + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\begin{pmatrix}\frac{\alpha}{s_{\hat{k}}} & \frac{\beta}{s_{\hat{h}}}\end{pmatrix}$,

$\boldsymbol{A}_1(t) = (\mu+\delta)\begin{pmatrix}\frac{\alpha}{s_{\hat{k}}}-1 & \frac{\alpha}{s_{\hat{h}}} \\ \frac{\alpha}{s_{\hat{k}}} & \frac{\alpha}{s_{\hat{h}}}-1\end{pmatrix}$, $\boldsymbol{A}_2(t) = \left(\mu + \frac{1+\phi-\theta}{1-\theta}\cdot\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\begin{pmatrix}\frac{\alpha}{s_{\hat{k}}}-1 & \frac{\alpha}{s_{\hat{h}}} \\ \frac{\alpha}{s_{\hat{k}}} & \frac{\alpha}{s_{\hat{h}}}-1\end{pmatrix}$, $\boldsymbol{M}_i^j = \left(\boldsymbol{A} - \frac{\gamma}{2}\boldsymbol{I} - \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}^j\right)_i$,

$\boldsymbol{A}_3(t) = \left(\mu + \frac{a(1-\xi)}{1+\xi(e^{at}-1)} + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\begin{pmatrix}\frac{\alpha}{s_{\hat{k}}}-1 & \frac{\alpha}{s_{\hat{h}}} \\ \frac{\alpha}{s_{\hat{k}}} & \frac{\alpha}{s_{\hat{h}}}-1\end{pmatrix}$ *and* $\boldsymbol{\zeta}_i^j(t) = \int_{t_0}^{t} e^{\boldsymbol{M}_i^j(t-\tau)}(1 \quad 1)^T\omega_i(t)e^{-\frac{\gamma}{2}t}d\tau$.

[9] *Unless otherwise stated, we have usually used* $\rho = 135$, $\alpha = 0.33$, $\beta = 0.1$, $s_{\hat{k}} = 0.285$, $s_{\hat{h}} = 0.0475$, $a = 0.0325$, $n = 0.02$, $\hat{k}_0 = 500$, $\hat{h}_0 = 100$, $\gamma = 0.045$, $\mu = 0.05$, $A_0 = 1$, $\phi = 0.8$, $\theta = 0.6$, $\delta = 0.042$ *and* $\xi = 0.05$.

$\sigma = 0$	*the population (labour) grows exponentially at its natural growth rate* n *over time.*
$\sigma = 1$	*the population (labour) will be static and not growing at all or growing at a 0.00% over time.*
$\sigma > 1$	*the population (labour) will be declining or growing at a negative rate over time.*
$0 < \sigma < 1$	*the population (labour) will be growing but at a reducing rate over time.*

For $\sigma > 0$, *the population growth dynamics is logistic. We assume that the natural growth rate of population,* n, *is* $0 < n < 1$.

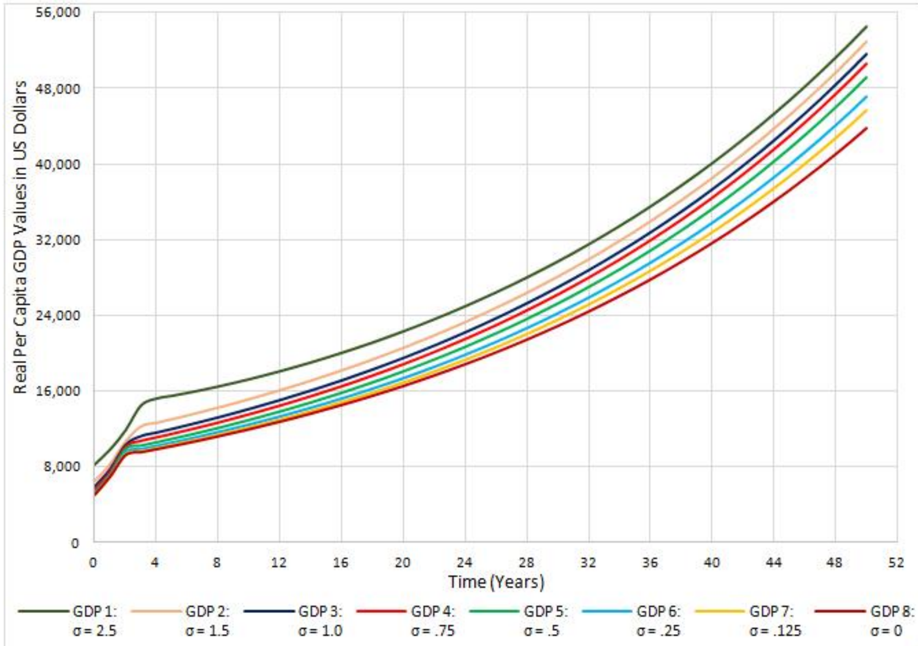

Fig. 4.1. Real per capita GDP trajectories for varying values of σ (With Tech)

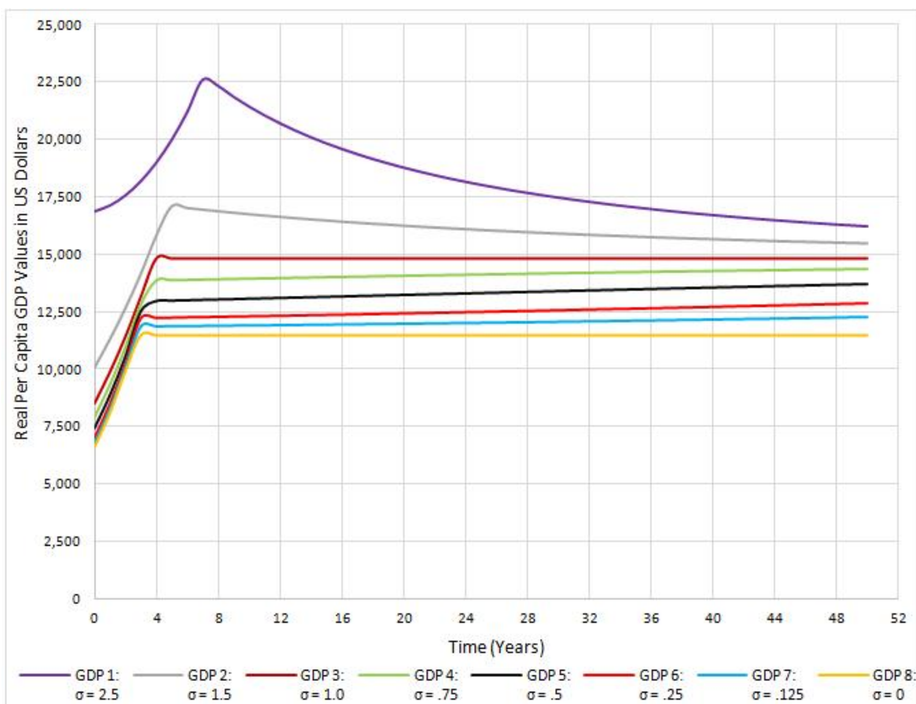

Fig. 4.2. Real per capita GDP trajectories for varying values of σ (No Tech)

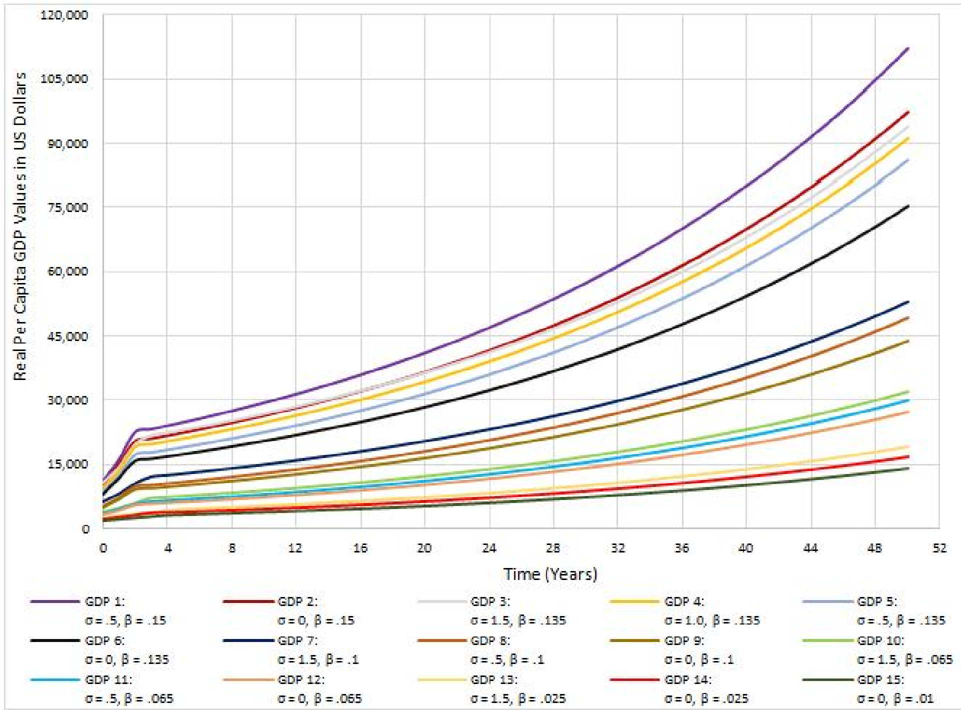

Fig. 4.3. Real per capita GDP trajectories for varying values of σ and β (With Tech)

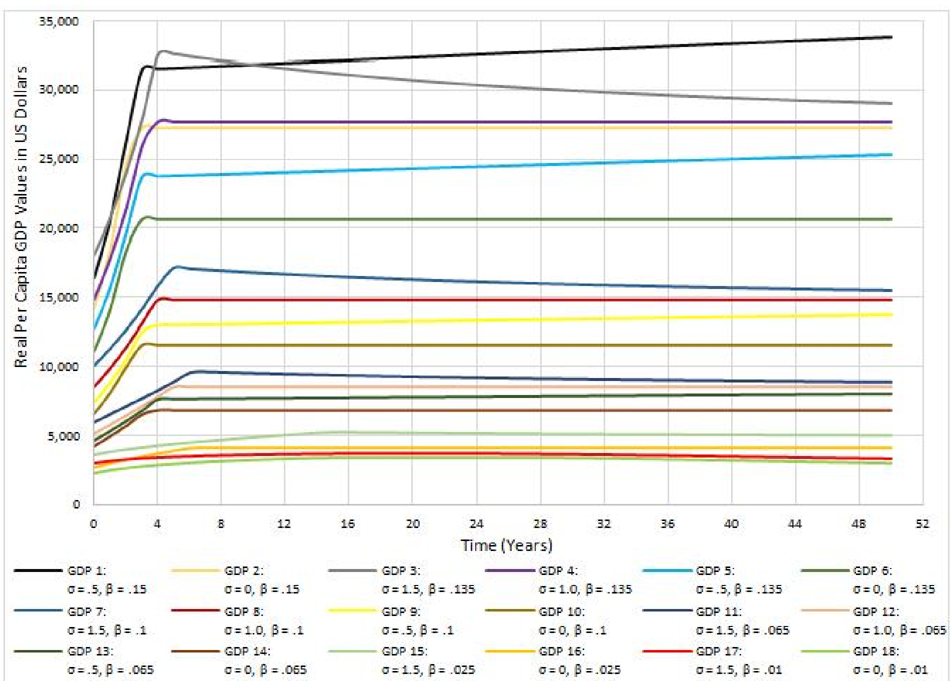

Fig. 4.4. Real per capita GDP trajectories for varying values of σ and β (No Tech)

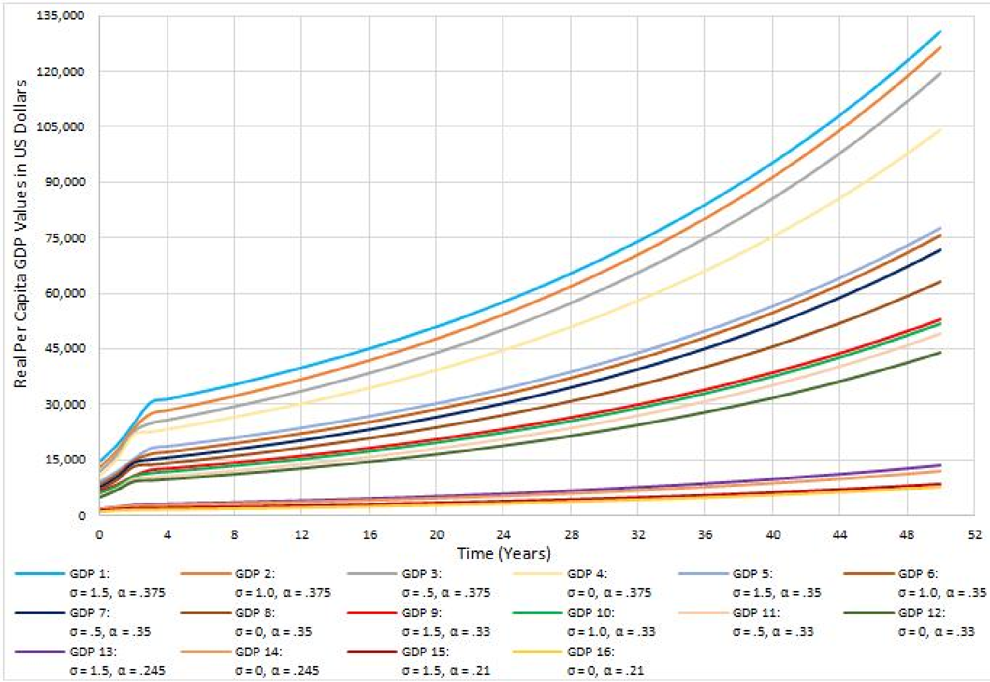

Fig. 4.5. Real per capita GDP trajectories for varying values of σ and α (With Tech)

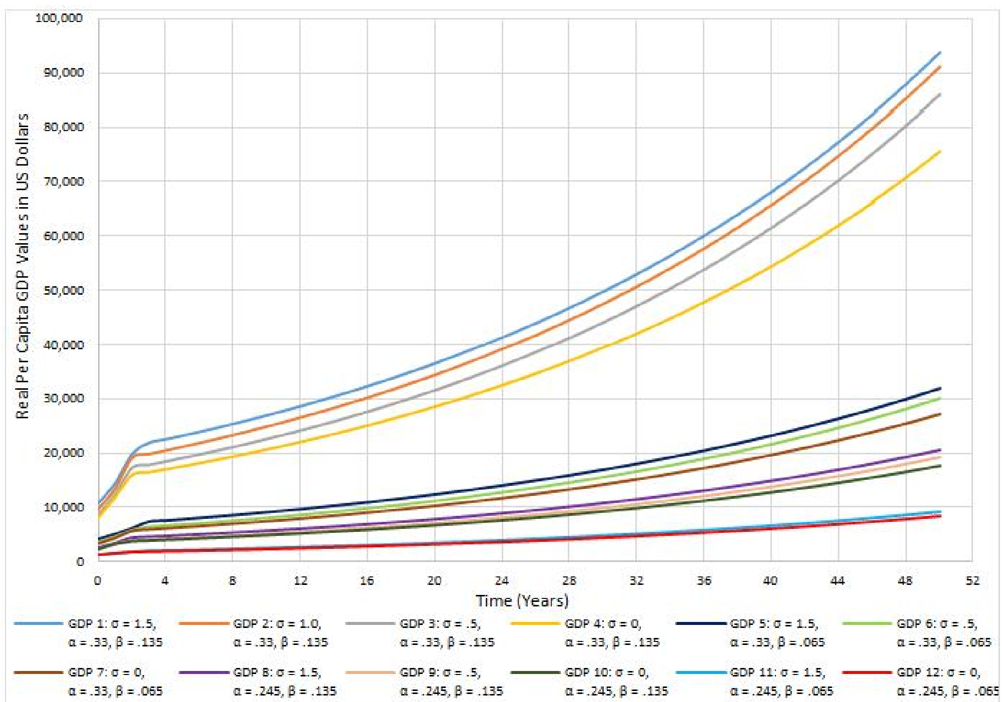

Fig. 4.6. Real per capita GDP trajectories for varying values of σ, β and α (With Tech)

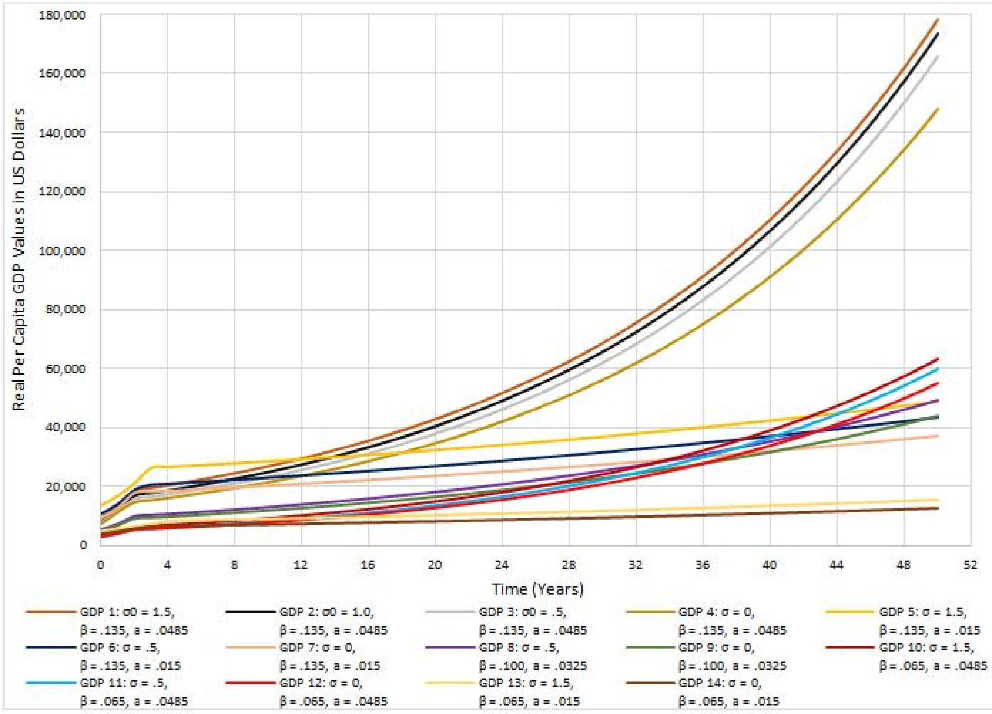

Fig. 4.7. Real per capita GDP trajectories for varying values of σ, β and a

It can be inferred from the above figures, as well as those beneath, except for plots generated under R & D technology (and to some extent, n), that as the population growth vary from purely exponential (when $\sigma = 0$) to strongly logistic (increasing σ values) real per capita income assumes higher trajectories. In each set of paths, the lowest corresponds to the case $\sigma = 0$. Higher bundles of trajectories show paths for increasing values of the parameter being varied (besides the parameter σ).

The results, and all the plots provided, especially Figs. 4.7, 4.11, and 4.14, illustrate clearly, as in [2], that $\frac{\partial y}{\partial a} > 0$, for all $t \geq 0$. Hence, increasingly higher values of a generate progressively higher $y(t)$ trajectories over time. As per Fig. 4.14, $y(t)$ takes a dip after the equilibrium when $a < 0$. It may initially rise though. Whenever $a = 0$, $y(t)$ initially rises over time, becomes flat after the equilibrium is attained, if $\sigma = 0$ or $\sigma = 1$. It however continues to rise, but very steadily, whenever $0 < \sigma < 1$. When $a = 0$, $y(t)$ drops after reaching the equilibrium point if $\sigma > 1$.

In the same way, the plots confirm that increasingly higher values of β generate correspondingly higher trajectories of $y(t)$, that is, $\frac{\partial y}{\partial \beta} > 0$, for $0 < \sigma < 1$, for all $t \geq 0$. However, $\frac{\partial y}{\partial \beta} < 0$, when $\sigma > 0$, for all $t \geq 0$, but $\frac{\partial y}{\partial \beta} = 0$ when $\sigma = 0$ or $\sigma = 1$. Figs. 4.3, 4.4, 4.6 and 4.7, as well as Fig. 4.10 attest to this. These also hold in respect of α. Fig. 4.5 and Fig. 4.6 above exemplify this.

Fig. 4.8 to Fig. 4.11 confirm that increasingly higher values of $s_{\hat{h}}$ generate progressively higher time-performance of $y(t)$, and so does this equally apply to $s_{\hat{k}}$. The converse is equally true in both cases. Thus $\frac{\partial y}{\partial s_{\hat{h}}} > 0$ and $\frac{\partial y}{\partial s_{\hat{k}}} > 0$, for all $t \geq 0$. (This also applies to ρ.) However, increasingly higher values of $s_{\hat{k}}$, as well as that of $s_{\hat{h}}$, generate higher time-trajectories of $y(t)$, but at a decreasing rate.

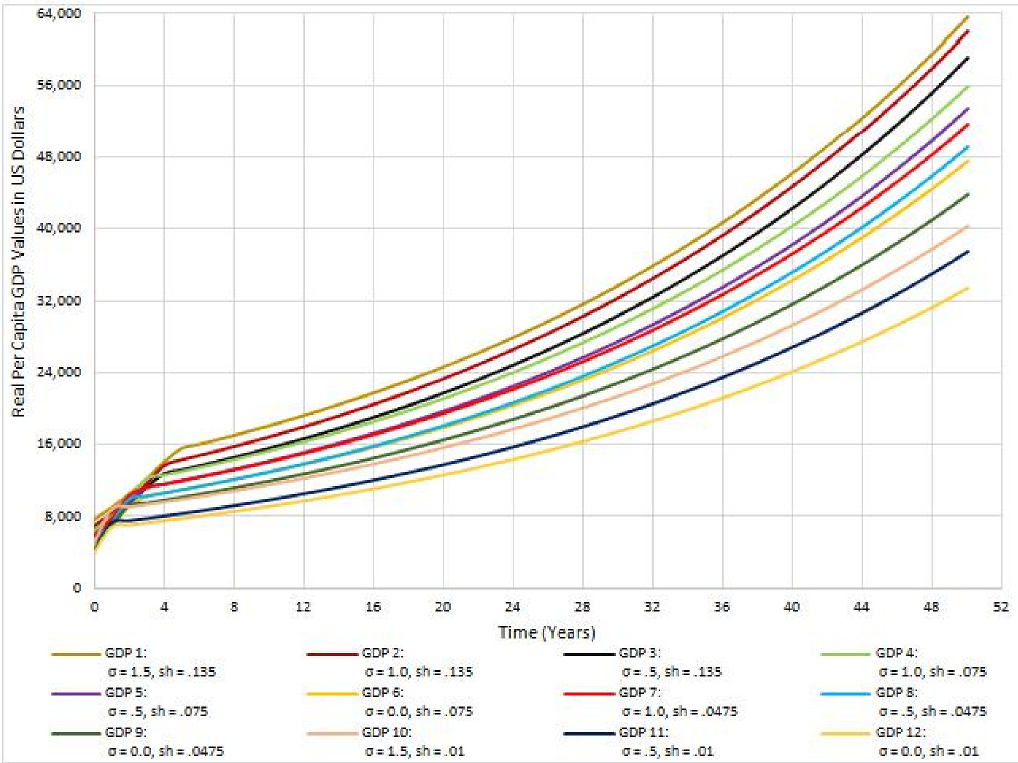

Fig. 4.8. Real per capita GDP trajectories for varying values of σ and s_h (With Tech)

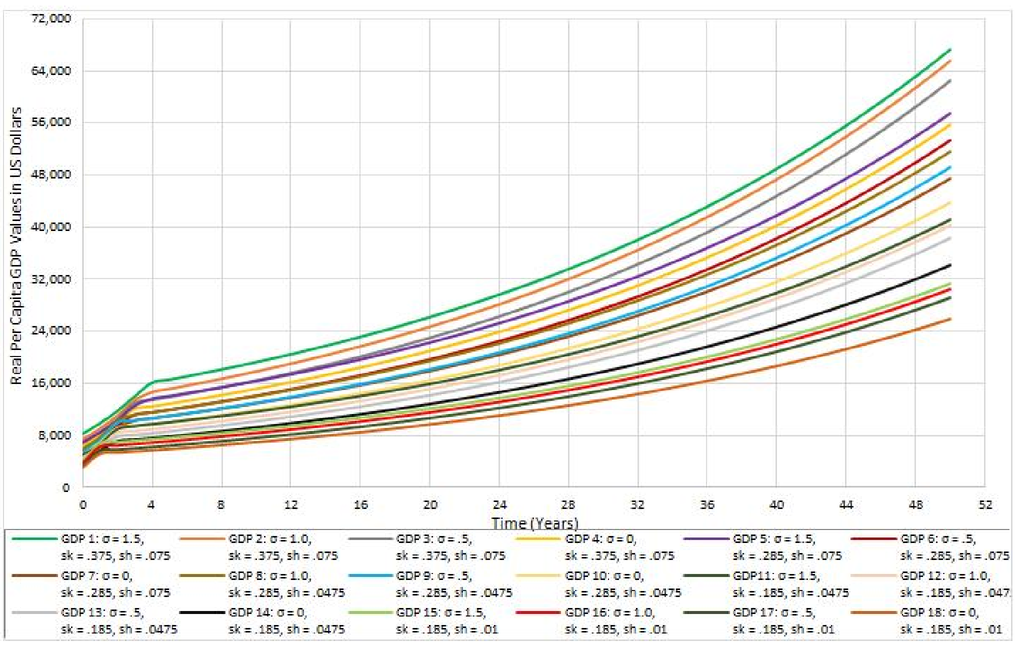

Fig. 4.9. Real per capita GDP trajectories for varying values of σ, s_h and s_k (With Tech)

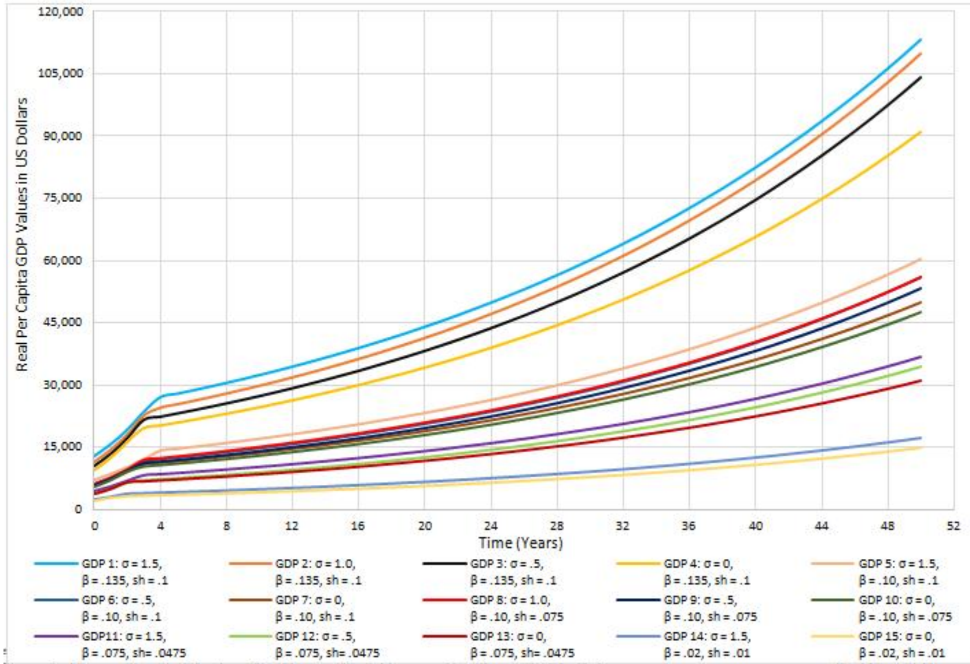

Fig. 4.10. Real per capita GDP trajectories for varying values of σ, s_h and β (With Tech)

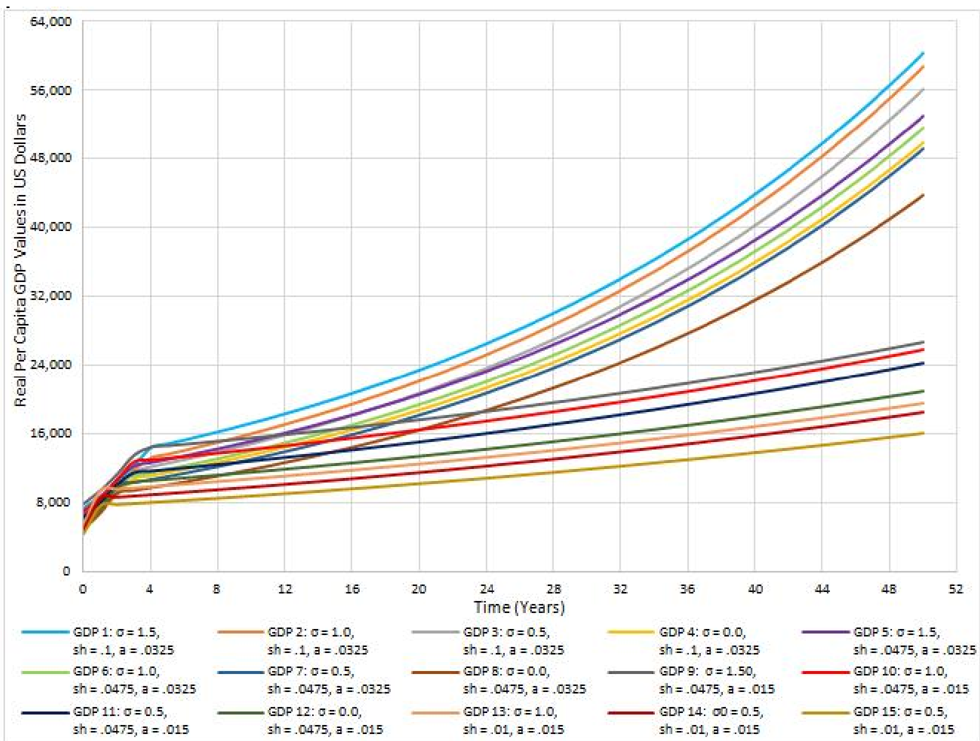

Fig. 4.11. Real per capita GDP trajectories for varying values of σ, s_h and a

The results further suggest that when $\sigma = 0$, the values of n inversely impact on the time-values of $y(t)$,[10] and thus higher values of n translate into lower trajectories of $y(t)$, and vice versa. Hence we have $\frac{\partial y}{\partial n} < 0$, when $\sigma = 0$, for all $t \geq 0$. Fig. 4.12 and Fig. 4.13 clearly illustrate this, and same when $\sigma > 1$. However, when $0 < \sigma < 1$, the effect of n on $y(t)$ is positive. The above referenced figures attest to this. (Clearly, $\frac{\partial y}{\partial \mu} < 0$, for all $t \geq 0$. Each of $\hat{k}_0$, $\hat{h}_0$ and γ has neutral effect on $y(t)$ over time.)

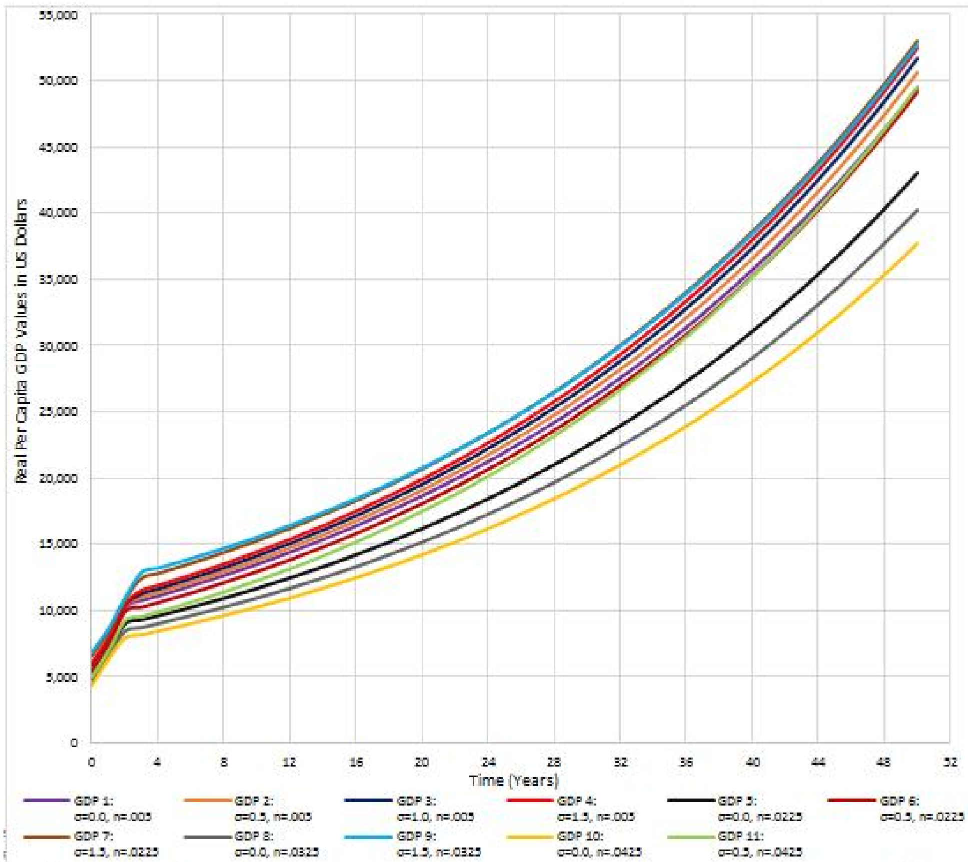

Fig. 4.12. Real per capita GDP trajectories for varying values of σ and n (With Tech)

4.2 Systems with non-constant technological growth dynamics

Similar to the analyses in [2], the effects of α, β, $s_{\hat{k}}$, and $s_{\hat{h}}$ (as well as those of ρ, $\hat{k}_0$, γ and μ) on the performance of $y(t)$ are largely the same as discussed earlier. As suspected earlier, the lower the value of n, the higher the growth prospects, and hence, the time-performance of $y(t)$ in the system with the modified residual technology, $A_1(t)$. That is $\frac{\partial y}{\partial n} < 0$, when $0 \leq \sigma < 1$. But $\frac{\partial y}{\partial n} > 0$, when $\sigma > 1$, and $\frac{\partial y}{\partial n} = 0$ when $\sigma = 1$. The very contrary is the truth when the modified R & D technological process, $A_2(t)$, is at play. The impact of n on $y(t)$ when $A_3(t)$ is at play is same as discussed in Section 4.1.[11]

[10] *Source data used is based on World Bank socio-economic data on countries [46]. Trajectories are not assigned to any economies due principally to the simulation analyses and generality of work done, which are of utmost importance here.*
[11] *Resid, R & D and Logist in Fig. 4.15 respectively denote the residual, R & D and logistic technological processes.*

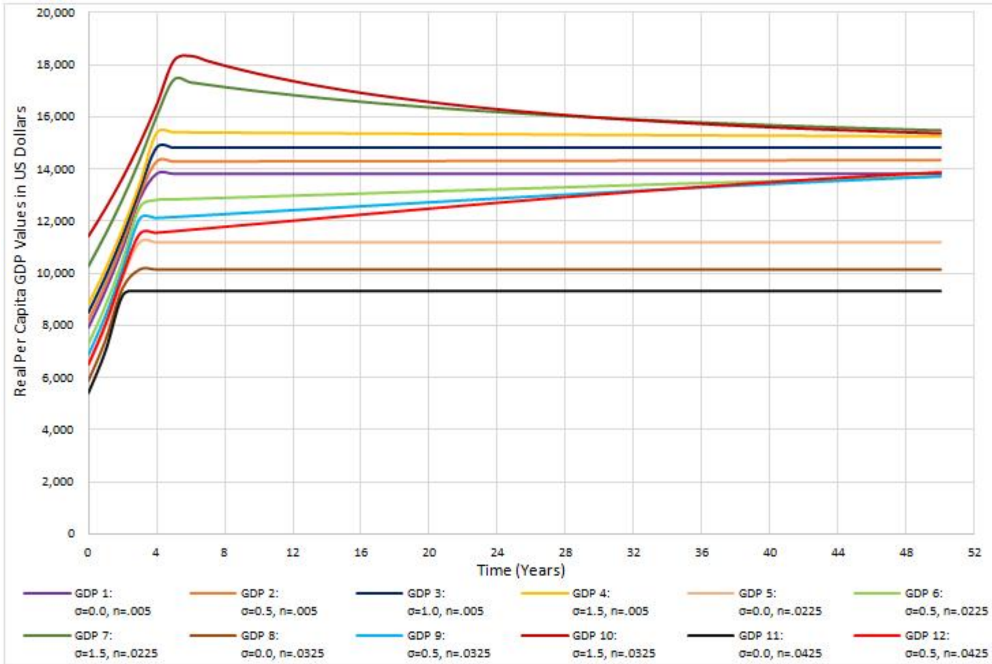

Fig. 4.13. Real per capita GDP trajectories for varying values of σ and n (No Tech)

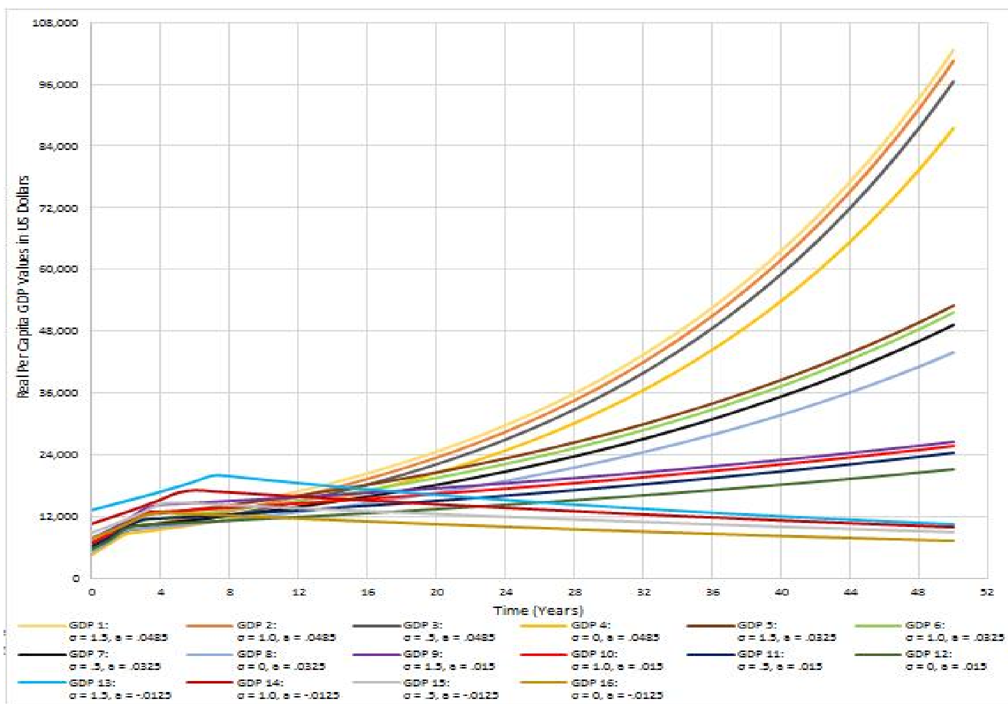

Fig. 4.14. Real per capita GDP trajectories for varying values of σ and a

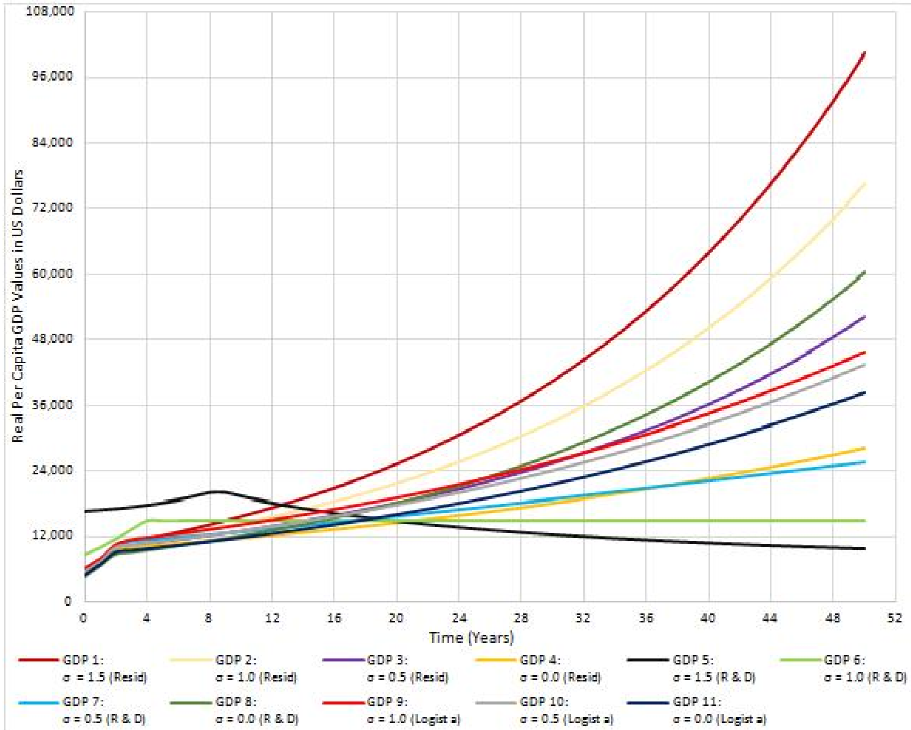

Fig. 4.15. Real per capita GDP trajectories for various Tech Processes and values of σ

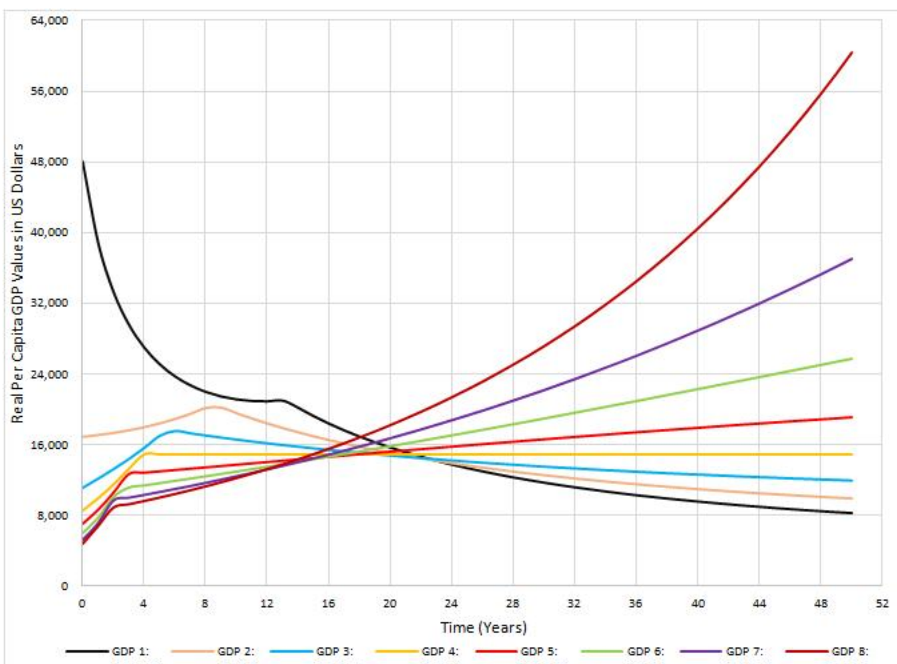

Fig. 4.16. Real per capita GDP trajectories for various values of σ Under R & D Technology

In systems underpinned by R & D, the performance of $y(t)$ is highest when $\sigma = 0$, and drastically slows down with increasing values of σ. See Figs. 4.15 and 4.16. The growth prospects in $y(t)$ are much enhanced for increasing values of ϕ and θ, especially the latter, for $0 < \sigma < 1$. But the effect of σ on $y(t)$ in systems with technological processes $A_1(t)$ and $A_3(t)$ is similar in nature to its effect on the systems with constant technological growth. Also, systems with the technology $A_3(t)$ grow fastest, and hence, generate the highest time-trajectory when $\xi = 0$, worsening with increasing values of ξ.

In the system with technological process $A_1(t)$, $y(t)$ grows much faster at a constant rate δ whenever $n = 0$ or $\sigma = 1$, and much more (no longer constant), when $\sigma > 1$ and $0 < n < 1$, except that $\sigma > 1$ is not desirable, per reasons assigned earlier. Anytime $\delta - n > 0$, the long term performance in $y(t)$ is quite good. Interestingly, $A_2(t)$ is much suited to economies whose populations grow exponentially, but the irony is that they scarcely seize this opportunity. Fig. 4.15 above illustrates the effects of the technological processes $A_i(t)$, for $i = 1, 2, 3$, on the performance of $y(t)$ over time.

5 Conclusions

The models built in the above are generally stable in the locality of their non-trivial equilibrium points, whenever $0 \leq \sigma \leq 1$. They are each locally controllable and observable. Consequently, the models' solutions are attainable and reachable, and that bounded inputs always induce bounded outputs. Thus as expected, their predictions are plausible and realistic. Most importantly, these results also hold true in the generalized N-factor aggregate production functions, and not limited to Cobb-Douglas forms.

Furthermore, it is also found from the comparative analyses of the results, in confirmation and extension of what pertain in [2], that:

1) Under the framework of R & D technology, economies with exponential population growth consistently perform better than those with logistic population growth in the long run. The more logistic the population growth is the worse the economic performance over time, and vice versa.[12]
2) On the contrary, in any other case excluding R & D technology, economies with exponential population growth consistently perform worse than those underlain by logistic population growth. The more logistic the population growth is the better the economic performance, and vice versa.
3) Higher technological growth is also found to be an excellent tool for rapid economic growth, and with this, it is clear from the simulation graphs that a lower income economy, over a time, can surpass that of a higher income economy whose technological growth is much less.
4) The inclusion of human capital and additional (not labour) factors of production generates greater time-performance of real per capita income. Thus added to technology, the capitalization of labour and the advent of other factors of production induce better economic performance.
5) The population dynamics parameter, σ, largely dictates, how most of the other parameters, and the technological processes, impact on the time-performance of real per capita income.
6) Given the caveat on σ, its tolerable domain is $0 \leq \sigma \leq 1$. The border line value $\sigma = 1$ may not be advisable given the caution on σ.
7) Generally, $0 < \sigma \leq 1$ is preferable for high time-performance of real per capita income, except with the residual technology where $\frac{\partial y}{\partial n} < 0$, whenever $0 \leq \sigma < 1$.
8) An economy with high natural population growth which adopts the R & D technological process or re-tunes the population dynamics into logistic experiences higher economic performance than the one that starts any of the above programmes from a relatively lower population growth rate.

Point (1) in the conclusion runs contrary to generally acknowledged economic theory which is supported by empirical evidence, re-enforced by point (2). This seemingly controversial result can be subtly adduced from

[12] *The rapid and incredible transformation of the S. Korean, Singaporean, Hong Kong, Malaysian and the Chinese economies, especially the onset of each economy's transformation until recent past, may be partially explained in the framework of this phenomenon.*

the Boserup's and Jone's theorems stated earlier. The above conclusion suggests that if developing economies could take the bold step and embrace R & D, they stand to gain more than from the population control measures preached by development partners. Moreover, developed economies should be mindful not to develop at the expense of running their populations into extinction.

Competing Interests

Authors have declared that no competing interests exist.

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