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On Approximate Solution of Fractional Differential Equations with Variable Coefficients

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Authors' contributions

This work was carried out in collaboration between both authors. Author HK designed the study, performed the computational analysis, and wrote the first draft of the manuscript. Author RAK managed the analysis of the study and literature searches and supervised the study. Both authors read and approved the final manuscript.

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Abstract

In this paper we study some interesting properties of shifted Jacobi polynomials and based on these properties a new operational matrix is derived. The new matrix is then used along with some previous results to provide a theoretical treatment to approximate the solution of fractional differential equations with variable coefficients. The scheme is then extended to solve coupled system of fractional differential equations with variable coefficients. The scheme is simple and provides a very high accurate estimate of solution. The accuracy of the scheme is shown with some test problems. The results are displayed graphically. We use MatLab to carry out the necessary calculation.

Keywords: Jacobi polynomials; operational matrices; fractional order differential equations; coupled system; Matlab.

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1 Introduction

Differential equations are of basic importance for modeling various engineering phenomena like Cascades, Compartment Analysis, Pond pollution, Home heating, Chemostats, Microorganism culturing, Nutrient flow in an aquarium, Biomass transfer, Forecasting prices, Electrical network, Earthquake effects on buildings and many more see for example [1]-[4]. After the discovery of fractional calculus it is rather shown that fractional order differential equations (FDEs) can well model the phenomena under consideration as compare to integer order differential equations. As usual to every advantage there is also some disadvantage, the same case occur when working with FDEs, the exact analytic solution is in most cases unavailable and in some cases this task become impossible. The reason to this difficulty is the high computational complexities of fractional calculus. The basic need in the current field of fractional calculus is to establish efficient numerical schemes, which are not only accurate and efficient but also simple such that a new researcher in the field can easily understand it.

In this context we extend a new algorithm for the numerical solutions of FDEs of the form

$$\frac{\partial^\sigma Y(x)}{\partial t^\sigma} = \sum_{i=1}^{n} \phi_i(x)\frac{\partial^i Y(x)}{\partial x^i} + \phi_0(x)Y(x) + f(x),$$

with some given initial conditions $Y^i(0) = a_i$, $i = 0, 1, \cdots n$. Where all a_i are real constants, $n < \sigma \leq n+1$, $x \in [0, \eta]$, $Y(x)$ is the unknown solution to be determined, $f(x)$ is the given source term and $\phi_i(x)$ for $i = 0, 1 \cdots n$ are coefficients depends on x and are well defined on $[0, \eta]$. The proposed method is then extended to investigate the numerical solutions of coupled system of FDEs of the form

$$\frac{\partial^\sigma Y(x)}{\partial x^\sigma} = \sum_{i=1}^{n} \phi_i(x)\frac{\partial^i Y(x)}{\partial x^i} + \sum_{i=1}^{n} \psi_i(x)\frac{\partial^i Z(x)}{\partial x^i} + \phi_0(x)Y(x) + \psi_0(x)Z(x) + f(x),$$
$$\frac{\partial^\sigma Z(x)}{\partial x^\sigma} = \sum_{i=1}^{n} \varphi_i(x)\frac{\partial^i Y(x)}{\partial x^i} + \sum_{i=1}^{n} \varrho_i(x)\frac{\partial^i Z(x)}{\partial x^i} + \varphi_0(x)Y(x) + \varrho_0(x)Z(x) + g(x),$$

with initial conditions $Y^i(0) = a_i$, $Z^i(0) = b_i$, $i = 0, 1, 2 \cdots n$. $n < \sigma \leq n+1$, a_i and b_i are real constant, $\phi_i(x), \psi_i(x), \varphi_i(x)$ and $\varrho_i(x)$ are given variable coefficients and are continuously differentiable and well defined on $[0, \eta]$. $Y(x)$ and $Z(x)$ are the unknown solutions to be determined and $f(x)$ and $g(x)$ are the given source terms. These types of equations are very important in modeling many real world problem, see for example [4]-[6]. In literature a huge amount of work is devoted to the study of such type of equations. In [7] A. Monje, use these equations to model the behavior of the immersed plate. Also in [4] such type of equation is used to model $PD^\nu-$controler, and it is proved that the fractional case can provide a more real insight in the phenomena rather than integer order. The equations governing these phenomena is just a special case of this generalized class of equations. Yi Chen et. al. [8] provides sufficient conditions under which the solution of the problem exists. They use Leggett-Williams fixed point theorem to prove the existence of positive solutions of the corresponding problem.

A huge amount of work is also devoted to find the solution of such types of problems. Recently [9] Yildiray Keskin proposed a new technique based of generalized Taylor polynomials for the numerical solution of such types of equations. More recently J. Liu et al [10] use the Legendre spectral Tau method to obtain the solution of fractional order partial differential equation with variable coefficients and many more. Some recent results in which orthogonal polynomials are applied to scientific problems can be found in [11]-[19].

Among others one of the most commonly used method for the solution of FDEs is the operational matrix technique, see for example [20]-[30] and the references quoted there. These operational matrices are based on various orthogonal polynomials and wavelets. A deep insight in the method shows that the method is really very simple and accurate. But up to now to the best of our knowledge the method is only used to approximate the solutions of differential equations, partial differential equations (including fractional order) only with constant coefficients. However the variable coefficients are solved with orthogonal polynomials in [31],[32]. For more study on spectral approximation and fractional calculus we refer the reader to study [33]-[35]. Some interested results related to approximation properties can be found in [36]-[39].

In this paper we make an attempt to generalize the operational matrix technique to solve FDEs and Coupled system of FDEs with variable coefficients under some appropriate initial conditions. The method is based on shifted Jacobi polynomials. By using some of the interesting properties of shifted Jacobi polynomials we develop a new operational matrix having very interesting properties. The new matrix can be named as Khalil matrix (K. matrix). The K. matrix allows us to convert the corresponding system to a system of easily solvable algebraic equations. From the analysis and numerical experiments we observe that the method is highly efficient for solving such problems.

The rest of the article is organized as follows : In section 2, we provide some preliminaries of fractional calculus, Jacobi polynomials and some basic results from approximation theory. In section 3 we derive K. Matrix and recall some other operational matrices of integration and differentiation for shifted Jacobi polynomials. In section 4, K. Matrix is used to establish a new scheme for solution of a generalized class of FDEs with variable coefficients and coupled FDEs with variable coefficients. In section 5, some numerical experiments is performed to show the efficiency of the new technique. The last section is devoted to a short conclusion.

2 Preliminaries

In this section we recall some basic definitions and concepts from open literature which are of basic importance in further development in this paper.

Definition 2.1. [40]-[41] According to Riemann-Liouville the fractional order integral of order $\alpha \in \mathbb{R}_+$ of a function $\phi \in (L^1[a,b],\mathbb{R})$ on interval $[a,b] \subset \mathbb{R}$, is defined by

$$\mathcal{I}_{a+}^{\alpha}\phi(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1}\phi(s)ds, \tag{2.1}$$

provided that the integral on right hand side exists.

Definition 2.2. For a given function $\phi(x) \in C^n[a,b]$, the Caputo fractional order derivative of order α is defined as

$$D^{\alpha}\phi(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{\phi^{(n)}(s)}{(x-s)^{\alpha+1-n}}ds, \ n-1 \le \alpha < n \ , n \in N, \tag{2.2}$$

provided that the right side is point wise defined on (a,∞), where $n = [\alpha]+1$.

From (2.1),(2.2) it is easily deducted that

$$D^{\alpha}x^k = \frac{\Gamma(1+k)}{\Gamma(1+k-\alpha)}x^{k-\alpha}, \quad I^{\alpha}x^k = \frac{\Gamma(1+k)}{\Gamma(1+k+\alpha)}x^{k+\alpha}, \tag{2.3}$$

and $D^{\alpha}C = 0$, for a constant C.

2.1 The shifted jacobi polynomials

The well known shifted Jacobi Polynomials are defined on $[0, \eta]$, having two parameter α and β. They are defined by the following relations

$$P_{\eta,i}^{(\alpha,\beta)}(t) = \sum_{k=0}^{i} \mho_{(i,k)}^{(\alpha,\beta)} x^k, i = 0, 1, 2, 3..., \tag{2.4}$$

where

$$\mho_{(i,k)}^{(\alpha,\beta)} = \frac{(-1)^{i-k}\Gamma(i+\beta+1)\Gamma(i+k+\alpha+\beta+1)}{\Gamma(k+\beta+1)\Gamma(i+\alpha+\beta+1)(i-k)!k!\eta^k}. \tag{2.5}$$

The orthogonality condition is

$$\int_0^{\eta} P_{\eta,i}^{(\alpha,\beta)}(x) P_{\eta,j}^{(\alpha,\beta)}(x) W_{\eta}^{(\alpha,\beta)}(x) dx = R_{\eta,j}^{(\alpha,\beta)} \delta_{i,j}, \tag{2.6}$$

where

$$W_{\eta}^{(\alpha,\beta)}(x) = (\eta - x)^{\alpha} x^{\beta}, and \quad R_{\eta,j}^{(\alpha,\beta)} = \frac{\eta^{\alpha+\beta+1}\Gamma(j+\alpha+1)\Gamma(j+\beta+1)}{(2j+\alpha+\beta+1)\Gamma(j+1)\Gamma(j+\alpha+\beta+1)}. \tag{2.7}$$

$W_{\eta}^{(\alpha,\beta)}(x)$ is called the weight function.

Any function $v(x)$ square integrable in $[0, \eta]$ can be approximated by shifted Jacobi polynomials as follows

$$v(x) \simeq \sum_{a=0}^{m} c_a P_{\eta,j}^{(\alpha,\beta)}(x), \tag{2.8}$$

as $m \to \infty$ the approximation becomes equal to the exact function. By the use of (2.6) and (2.7) we can easily calculate the coefficient c_a. We can also write (2.8) in vector form as

$$v(x) \simeq H_M^T \hat{\Psi}_M(x), \tag{2.9}$$

where $M = m + 1$, H_M is the coefficient vector and $\hat{\Psi}_M(x)$ is M terms vector function. The following lemma is very important for our further analysis.

Lemma 2.1. *The integral of a function f on the domain $[0, \eta]$ where f is the product of weight function with any three Jacobi polynomials is a constant and the value of that constant is $\digamma_{(l,m,n)}^{(i,j,k)}$ ie*

$$\int_0^{\eta} W_{\eta}^{(\alpha,\beta)}(x) P_{\eta,i}^{(\alpha,\beta)}(x) P_{\eta,j}^{(\alpha,\beta)}(x) P_{\eta,k}^{(\alpha,\beta)}(x) dx = \digamma_{(l,m,n)}^{(i,j,k)}, \tag{2.10}$$

where $\digamma_{(l,m,n)}^{(i,j,k)} = \sum_{l=0}^{i} \sum_{m=0}^{j} \sum_{n=0}^{k} \mho_{(i,l)}^{(\alpha,\beta)} \mho_{(j,m)}^{(\alpha,\beta)} \mho_{(k,n)}^{(\alpha,\beta)} \Upsilon_{(l,m,n)}$. $\mho_{(.,.)}^{(\alpha,\beta)}$ *are as defined in* (2.5) *and*

$$\Upsilon_{(l,m,n)} = \frac{\Gamma(l+m+n+\beta+1)\Gamma(\alpha+1)\eta^{(l+m+n+\alpha+\beta+1)}}{\Gamma(l+m+n+\alpha+\beta+1)}.$$

Proof. Consider

$$\int_0^{\eta} W_{\eta}^{(\alpha,\beta)}(x) P_{\eta,i}^{(\alpha,\beta)}(x) P_{\eta,j}^{(\alpha,\beta)}(x) P_{\eta,k}^{(\alpha,\beta)}(x) dx = \sum_{l=0}^{i} \mho_{(i,l)}^{(\alpha,\beta)} \sum_{m=0}^{j} \mho_{(j,m)}^{(\alpha,\beta)} \sum_{n=0}^{k} \mho_{(k,n)}^{(\alpha,\beta)} \int_0^{\eta} x^{(l+m+n+\beta)} (\eta - x)^{\alpha} dx. \tag{2.11}$$

By calculating the integral in the above equation we get

$$\int_0^{\eta} W_{\eta}^{(\alpha,\beta)}(x) P_{\eta,i}^{(\alpha,\beta)}(x) P_{\eta,j}^{(\alpha,\beta)}(x) P_{\eta,k}^{(\alpha,\beta)}(x) dx = \sum_{l=0}^{i} \sum_{m=0}^{j} \sum_{n=0}^{k} \mho_{(i,l)}^{(\alpha,\beta)} \mho_{(j,m)}^{(\alpha,\beta)} \mho_{(k,n)}^{(\alpha,\beta)} \Upsilon_{(l,m,n)}. \tag{2.12}$$

Where$\Upsilon_{(l,m,n)} = \frac{\Gamma(l+m+n+\beta+1)\Gamma(\alpha+1)\eta^{(l+m+n+\alpha+\beta+1)}}{\Gamma(l+m+n+\alpha+\beta+1)}$. Representing the left side of (2.12) with $\digamma_{(l,m,n)}^{(i,j,k)}$ will complete the proof. □

2.2 Error analysis and spectral accuracy

For sufficiently smooth function $g(x) \in \Delta$, where $\Delta = [0, \eta]$. Consider $\prod_M(x)$ be the space of M terms Jacobi polynomials. Assume that $g_{(M)}(x)$ is its best approximation in $\prod_{(M)}(x)$. Then, for any polynomial $\hat{Q}_{(M)}(x)$ of degree $\leq M$ in variable x it follows that

$$\|g(x) - g_{(M)}(x)\|_2 \leq \|g(x) - \hat{Q}_{(M)}(x)\|_2. \tag{2.13}$$

The inequality (2.13) also holds if $\hat{Q}_{(M)}(x)$ is interpolating polynomial of the function g at point (x_i) where $x_i = i\frac{\eta}{M}$. Then by similar processes as in [42] we can easily get

$$\|g(x) - \hat{Q}_{(M)}(x)\|_2 \leq (C_1 \frac{1}{M^{M+1}}), \tag{2.14}$$

where $C_1 = \frac{1}{4}\max_{x\in[0,\eta]} |\frac{\partial^{M+1}}{\partial x^{M+1}} g(x)|$. By the arguments given in [43]-[42] we can easily prove the above equation. The spectral accuracy and decay of the expansion coefficients can be guaranteed by the following Lemma.

Lemma 2.2. *Let $g(x) \in \prod_M(x)$ and $g(x) = \sum_{k=0}^{m} c_k P_{\eta,k}^{(\alpha,\beta)}(x)$, then*

$$| c_k | \simeq \frac{C}{(\lambda_k)^m} \| g_{(m)} \|, \tag{2.15}$$

and

$$\| g(x) - \sum_{k=0}^{m} c_k P_{\eta,k}^{(\alpha,\beta)}(x) \|^2 = \sum_{k=m}^{\infty} \gamma_k c_k^2. \tag{2.16}$$

Where $\lambda_k = k(k + \alpha + \beta + 1)$, and $c_k = \frac{1}{R_{\eta,j}^{(\alpha,\beta)}} \int_0^1 y(x) P_{\eta,k}^{(\alpha,\beta)}(x) W_\eta^{(\alpha,\beta)}(x)dx$. C is a constant and m can be chosen in a way such that $y_{(2m)} \in \prod_M(x)$. Also we have the equality

$$g_{(m)} = \frac{1}{W_\eta^{(\alpha,\beta)}(x)} L g_{(m-1)}(x) = (\frac{L}{W_\eta^{(\alpha,\beta)}(x)})^m g(x).$$

Where L is the Sturm-Liouville operator, and $g_{(0)} = g(x)$.

Proof. For the proof of this Lemma we refer the reader to [44]. For more detailed study see [45]. □

The convergence of any function and its Jacobi spectral expansion solely depends on the power decay of the Jacobi spectral expansion coefficients. Consequently from above lemma, we conclude that if the function $g(x) \in C^\infty[0, 1]$, we recover spectral decay of the expansion coefficients i.e.,$|c_k|$ decays faster than any algebraic order of λ_k. This result is valid and independent of specific boundary conditions on $g(x)$.

3 Operational Matrices of Integration and Differentiation

The following Lemmas are important to establish our result.

Lemma 3.1. *Let $\Psi_M(x)$ be the function vector as defined in (2.9) then the fractional derivative of order σ of $\Psi_M(x)$ is generalized as $D^\sigma(\Psi_M(x)) \simeq G_{M\times M}^{\eta,\sigma}\Psi_M(x)$, where $G_{M\times M}^{\eta,\sigma}$ is the operational matrix of derivative of order σ and is defined as*

$$G_{M\times M}^{\eta,\sigma} = \begin{bmatrix} \Theta_{0,0,k} & \Theta_{0,1,k}, & \cdots & \Theta_{0,j,k} & \cdots & \Theta_{0,m,k} \\ \Theta_{1,0,k} & \Theta_{1,1,k} & \cdots & \Theta_{1,j,k} & \cdots & \Theta_{1,m,j} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Theta_{i,0,k} & \Theta_{i,1,k} & \cdots & \Theta_{i,j,k} & \cdots & \Theta_{i,m,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Theta_{m,0,k} & \Theta_{m,1,k} & \cdots & \Theta_{m,j,k} & \cdots & \Theta_{m,m,k} \end{bmatrix}. \tag{3.1}$$

Where

$$\Theta_{i,j,k} = \sum_{k=0}^{i} \Lambda_{i,k,\sigma} S_j, \quad \Lambda_{i,k,\sigma} = \frac{(-1)^{i-k}\Gamma(i+\beta+1)\Gamma(i+k+\alpha+\beta+1)\Gamma(1+k)}{\Gamma(k+\beta+1)\Gamma(i+\alpha+\beta+1)(i-k)!k!\Gamma(1+k-\sigma)\eta^k}. \tag{3.2}$$

$$S_j = \sum_{l=0}^{j} \frac{(-1)^{j-l}(2j+\alpha+\beta+1)\Gamma(j+1)\Gamma(j+l+\alpha+\beta+1)\Gamma(k-\sigma+l+\beta+1)\Gamma(\alpha+1)\eta^{\sigma}}{\Gamma(j+\alpha+1)\Gamma(l+\beta+1)(j-l)!l!\Gamma(k-\sigma+l+\beta+\alpha+2)}. \tag{3.3}$$

Also $\Theta_{i,j,k} = 0$ *if* $i < \sigma$.

Proof. The proof of this Lemma is discussed in details by E.H. Doha in [46]. □

Lemma 3.2. *Let* $\Psi_M(x)$ *be the function vector as defined in* (2.9) *then the fractional integral of order* γ *of* $\Psi_M(x)$ *is generalized as* $I^{\gamma}(\Psi_M(x)) \simeq H^{\eta,\gamma}_{M\times M}\Psi_M(x)$, *where* $H^{\eta,\gamma}_{M\times M}$ *is the operational matrix of integration of order* γ *and is defined as*

$$H^{\eta,\gamma}_{M\times M} = \begin{bmatrix} \Theta_{0,0,k} & \Theta_{0,1,k}, & \cdots & \Theta_{0,j,k} & \cdots & \Theta_{0,m,k} \\ \Theta_{1,0,k} & \Theta_{1,1,k} & \cdots & \Theta_{1,j,k} & \cdots & \Theta_{1,m,j} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Theta_{i,0,k} & \Theta_{i,1,k} & \cdots & \Theta_{i,j,k} & \cdots & \Theta_{i,m,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Theta_{m,0,k} & \Theta_{m,1,k} & \cdots & \Theta_{m,j,k} & \cdots & \Theta_{m,m,k} \end{bmatrix}. \tag{3.4}$$

Where

$$\Theta_{i,j,k} = \sum_{k=0}^{i} \Lambda_{i,k,\gamma} S_j, \quad \Lambda_{i,k,\gamma} = \frac{(-1)^{i-k}\Gamma(i+\beta+1)\Gamma(i+k+\alpha+\beta+1)\Gamma(1+k)}{\Gamma(k+\beta+1)\Gamma(i+\alpha+\beta+1)(i-k)!k!\Gamma(1+k+\gamma)\eta^k}, \tag{3.5}$$

$$S_j = \sum_{l=0}^{j} \frac{(-1)^{j-l}(2j+\alpha+\beta+1)\Gamma(j+1)\Gamma(j+l+\alpha+\beta+1)\Gamma(k+\gamma+l+\beta+1)\Gamma(\alpha+1)\eta^{\gamma}}{\Gamma(j+\alpha+1)\Gamma(l+\beta+1)(j-l)!l!\Gamma(k+\gamma+l+\beta+\alpha+2)}. \tag{3.6}$$

Proof. The proof of this Lemma is similar as Lemma 3.0.1. □

Lemma 3.3. *Let* $Y(x)$ *and* $\phi_n(x)$ *be any function defined on* $[0,\eta]$. *Then*

$$\phi_n(x)\frac{\partial^{\sigma}Y(x)}{\partial x^{\sigma}} = W_M^T K^{\sigma}_{\phi_n}\Psi_M(x). \tag{3.7}$$

Where W_M^T *is the Jacobi coefficient vector of* $Y(x)$ *as defined in* (2.9) *and*

$$K^{\sigma}_{\phi_n} = G^{\eta,\sigma}_{M\times M} J^{\eta,\phi_n}_{M\times M}. \tag{3.8}$$

The matrix $G^{\eta,\sigma}_{M\times M}$ *is the operational matrix of derivative as defined in Lemma 3.1 and*

$$J^{\eta,\phi_n}_{M\times M} = \begin{pmatrix} \Theta_{0,0} & \Theta_{0,1} & \cdots & \Theta_{0,s} & \cdots & \Theta_{0,m} \\ \Theta_{1,0} & \Theta_{1,1} & \cdots & \Theta_{1,s} & \cdots & \Theta_{1,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Theta_{r,0} & \Theta_{r,1} & \cdots & \Theta_{r,s} & \cdots & \Theta_{r,m} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \Theta_{m,0} & \Theta_{m,1} & \cdots & \Theta_{m,s} & \cdots & \Theta_{m,m} \end{pmatrix}. \tag{3.9}$$

Where

$$\Theta_{r,s} = \frac{1}{R^{(\alpha,\beta)}_{\eta,s}} \sum_{i=0}^{m} c_i F^{(i,r,s)}_{(l,m,n)}. \tag{3.10}$$

$R^{(\alpha,\beta)}_{\eta,s}$ *is as defines in* (2.7) *and* $c_i = \int_0^\eta \phi_n(x) P^{(\alpha,\beta)}_{\eta,i}(x) W^{(\alpha,\beta)}_{\eta}(x)dx$.

Proof. Consider $Y(x) \simeq W_M^T \Psi_M(x)$, then by the use of Lemma 3.0.1 we get

$$\phi_n(x)\frac{\partial^\sigma Y(x)}{\partial x^\sigma} = \phi_n(x) W_M^T G^{\eta,\sigma}_{M\times M} \Psi_M(x). \tag{3.11}$$

On rearranging we can write the above equation as

$$\phi_n(x)\frac{\partial^\sigma Y(x)}{\partial x^\sigma} = W_M^T G^{\eta,\sigma}_{M\times M} \overbrace{\Psi_M(x)}. \tag{3.12}$$

Where

$$\overbrace{\Psi_M(x)} = \begin{bmatrix} \phi_n(x)P^{(\alpha,\beta)}_{\eta,0}(x) & \phi_n(x)P^{(\alpha,\beta)}_{\eta,1}(x) & \cdots & \phi_n(x)P^{(\alpha,\beta)}_{\eta,r}(x) & \cdots & \phi_n(x)P^{(\alpha,\beta)}_{\eta,m}(x) \end{bmatrix}^T. \tag{3.13}$$

Now let

$$\phi_n(x) = \sum_{i=0}^{m} c_i P^{(\alpha,\beta)}_{\eta,i}(x). \tag{3.14}$$

Using (3.14) in (3.13) we can get

$$\overbrace{\Psi_M(x)} = \begin{bmatrix} \aleph_0(x) & \aleph_1(x) & \cdots & \aleph_r(x) & \cdots & \aleph_m(x) \end{bmatrix}^T, \tag{3.15}$$

where

$$\aleph_r(x) = \sum_{i=0}^{m} c_i P^{(\alpha,\beta)}_{\eta,i}(x) P^{(\alpha,\beta)}_{\eta,r}(x)\, , r = 0, 1\cdots m \tag{3.16}$$

Now consider the general term $\aleph_r(x)$, we can approximate it with Jacobi polynomials as

$$\aleph_r(x) = \sum_{s=0}^{m} d_s^r P^{(\alpha,\beta)}_{\eta,s}(x), \tag{3.17}$$

where

$$d_s^r = \frac{1}{R^{(\alpha,\beta)}_{\eta,s}} \int_0^\eta \aleph_r(x) W^{(\alpha,\beta)}_{\eta}(x) P^{(\alpha,\beta)}_{\eta,s}(x)dx. \tag{3.18}$$

Using (3.16) in (3.18) we get

$$d_s^r = \frac{1}{R^{(\alpha,\beta)}_{\eta,s}} \sum_{i=0}^{m} c_i \int_0^\eta W^{(\alpha,\beta)}_{\eta}(x) P^{(\alpha,\beta)}_{\eta,i}(x) P^{(\alpha,\beta)}_{\eta,r}(x) P^{(\alpha,\beta)}_{\eta,s}(x)dx. \tag{3.19}$$

Now using Lemma 2.2.1 we get

$$d_s^r = \frac{1}{R^{(\alpha,\beta)}_{\eta,s}} \sum_{i=0}^{m} c_i F^{(i,r,s)}_{(l,m,n)}. \tag{3.20}$$

Let suppose

$$\Theta_{r,s} = \frac{1}{R^{(\alpha,\beta)}_{\eta,s}} \sum_{i=0}^{m} c_i F^{(i,r,s)}_{(l,m,n)}. \tag{3.21}$$

Then repeating the procedure for $r = 0, 1, \cdots m$ and $s = 0, 1, \cdots m$ we can write

$$\begin{bmatrix} \aleph_0(x) \\ \aleph_1(x) \\ \cdots \\ \aleph_r(x) \\ \cdots \\ \aleph_m(x) \end{bmatrix} = \begin{bmatrix} \Theta_{0,0} & \Theta_{0,1} & \cdots & \Theta_{0,s} & \cdots & \Theta_{0,m} \\ \Theta_{1,0} & \Theta_{1,1} & \cdots & \Theta_{1,s} & \cdots & \Theta_{1,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Theta_{r,0} & \Theta_{r,1} & \cdots & \Theta_{r,s} & \cdots & \Theta_{r,m} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \Theta_{m,0} & \Theta_{m,1} & \cdots & \Theta_{m,s} & \cdots & \Theta_{m,m} \end{bmatrix} \begin{bmatrix} P_{\eta,0}^{(\alpha,\beta)}(x) \\ P_{\eta,1}^{(\alpha,\beta)}(x) \\ \vdots \\ P_{\eta,s}^{(\alpha,\beta)}(x) \\ \vdots \\ P_{\eta,m}^{(\alpha,\beta)}(x) \end{bmatrix}. \tag{3.22}$$

In simplified notation we can write the above equation as

$$\overbrace{\Psi_M(x)} = J_{M\times M}^{\eta,\phi_n} \Psi_M(x). \tag{3.23}$$

Using (3.23) in (3.12) we get

$$\phi_n(x) \frac{\partial^\sigma Y(x)}{\partial x^\sigma} = W_M^T G_{M\times M}^{\eta,\sigma} J_{M\times M}^{\eta,\phi_x} \Psi_M(x). \tag{3.24}$$

Let $G_{M\times M}^{\eta,\sigma} J_{M\times M}^{\eta,\phi_n} = K_{\phi_n}^\sigma$. Then we have $\phi_n(x) \frac{\partial^\sigma Y(x)}{\partial x^\sigma} = W_M^T K_{\phi_n}^\sigma \Psi_M(x)$. And hence the proof is complete. □

4 Application of the K-matrix

K-matrix is of basic importance in approximating the solutions of a wide and important class of fractional differential equations with variable coefficients. Here we show some of its application to a small class of fractional differential equations and coupled system of fractional differential equations.

4.1 FDEs with variable equations

Consider the following generalized class of FDEs with variable coefficients

$$\frac{\partial^\sigma Y(x)}{\partial x^\sigma} = \sum_{i=1}^{n} \phi_i(x) \frac{\partial^i Y(x)}{\partial x^i} + \phi_0(x) Y(x) + f(x), \tag{4.1}$$

with initial conditions $Y^i(0) = a_i$, $i = 0, 1, \cdots n$. Where a_i are all real constant, $n < \sigma \leq n + 1$, $x \in [0, \eta]$, $Y(x)$ is the solution to be determined, $f(x)$ is the given source term and $\phi_i(x)$ for $i = 0, 1 \cdots n$ are coefficients depends on x and are well defined on $[0, \eta]$. We seek the solution of the problem in terms of Jacobi polynomials such that

$$D^\sigma Y(x) = W_M^T \Psi_M(x). \tag{4.2}$$

Applying fractional integral of order σ and using the initial conditions we get

$$Y(x) - \sum_{s=0}^{n} x^s a_s = W_M^T H_{M\times M}^{\eta,\sigma} \Psi_M(x). \tag{4.3}$$

We can also write the above expression in the following form

$$Y(x) = W_M^T H_{M\times M}^{\eta,\sigma} \Psi_M(x) + F_1^T \Psi_M(x), \quad \text{where} \quad F_1^T \Psi_M(x) = \sum_{s=0}^{n} x^s a_s \tag{4.4}$$

On further simplification we get

$$Y(x) = \hat{W}_M^T \Psi_M(x), \quad \text{where} \quad \hat{W}_M^T = W_M^T H_{M\times M}^{\eta,\sigma} + F_1^T. \tag{4.5}$$

Using (4.5) along with Lemma 3.0.3 we can write

$$\phi_i(x)\frac{\partial^i Y(x)}{\partial x^i} = \hat{W}_M^T K_{\phi_i}^i \Psi_M(x), \quad \text{and} \quad \phi_0(x)Y(x) = \hat{W}_M^T K_{\phi_0}^0 \Psi_M(x). \tag{4.6}$$

Approximating $f(x) = F_2\Psi_M(x)$ and using along with (4.6) in (4.1) we get

$$W_M^T \Psi_M(x) = \sum_{i=1}^{n} \hat{W}_M^T K_{\phi_i}^i \Psi_M(x) + \hat{W}_M^T K_{\phi_0}^0 \Psi_M(x) + F_2\Psi_M(x). \tag{4.7}$$

On further simplification we get

$$W_M^T \Psi_M(x) = \hat{W}_M^T \sum_{i=0}^{n} K_{\phi_i}^i \Psi_M(x) + F_2\Psi_M(x). \tag{4.8}$$

We can also write it as $\{W_M^T - \hat{W}_M^T \sum_{i=0}^{n} K_{\phi_i}^i - F_2\}\Psi_M(x) = 0$. Canceling out the common term and using the value of $\hat{W}_M^T$ from (4.5) we get

$$\{W_M^T - W_M^T \sum_{i=0}^{n} H_{M\times M}^{\eta,\sigma} K_{\phi_i}^i - \sum_{i=0}^{n} F_1 K_{\phi_i}^i - F_2\} = 0. \tag{4.9}$$

Equation (4.9) is a generalized Lyapunov type matrix equation and can be easily solved with any computational software for the unknown vector W_M^T.Using the value of W_M^T in equation (4.4) we get approximate solution to the problem.

4.2 Coupled system of FDEs with variable equations

The K-Matrix also plays important role in the coupled system of FDEs.Consider the following system

$$\begin{aligned}
\frac{\partial^\sigma Y(x)}{\partial x^\sigma} &= \sum_{i=1}^{n} \phi_i(x)\frac{\partial^i Y(x)}{\partial x^i} + \sum_{i=1}^{n} \psi_i(x)\frac{\partial^i Z(x)}{\partial x^i} + \phi_0(x)Y(x) + \psi_0(x)Z(x) + f(x),\\
\frac{\partial^\sigma Z(x)}{\partial x^\sigma} &= \sum_{i=1}^{n} \varphi_i(x)\frac{\partial^i Y(x)}{\partial x^i} + \sum_{i=1}^{n} \varrho_i(x)\frac{\partial^i Z(x)}{\partial x^i} + \varphi_0(x)Y(x) + \varrho_0(x)Z(x) + g(x),
\end{aligned} \tag{4.10}$$

with initial conditions $Y^i(0) = a_i, \quad Z^i(0) = b_i, \quad i = 0, 1, 2 \cdots n$. We seek the solutions of the system in terms of shifted Jacobi polynomials such that

$$\frac{\partial^\sigma Y(x)}{\partial x^\sigma} = R_M^T \Psi_M(x), \qquad \frac{\partial^\sigma Z(x)}{\partial x^\sigma} = S_M^T \Psi_M(x). \tag{4.11}$$

Applying fractional integral of order σ on the corresponding equations and using initial conditions we get (4.11) as

$$Y(x) = \hat{R}_M^T \Psi_M(x), \qquad Z(x) = \hat{S}_M^T \Psi_M(x). \tag{4.12}$$

Where

$$\hat{R}_M^T = R_M^T H_{M\times M}^{\eta,\sigma} + F_1^T, \qquad \hat{S}_M^T = S_M^T H_{M\times M}^{\eta,\sigma} + F_2^T. \tag{4.13}$$

Note here that $F_1^T \Psi_M(x) = \sum_{i=0}^{n} a_i x^i$ and $F_2^T \Psi_M(x) = \sum_{i=0}^{n} b_i x^i$. Using equation (4.12) along with Lemma 3.0.3 we may write

$$\begin{aligned}
&\sum_{i=1}^{n} \phi_i(x)\frac{\partial^i Y(x)}{\partial x^i} = \hat{R}_M^T \sum_{i=1}^{n} K_{\phi_i}^i \Psi_M(x), && \sum_{i=1}^{n} \psi_i(x)\frac{\partial^i Z(x)}{\partial x^i} = \hat{S}_M^T \sum_{i=1}^{n} K_{\psi_i}^i \Psi_M(x),\\
&\phi_0(x)Y(x) = \hat{R}_M^T K_{\phi_o}^0 \Psi_M(x), && \psi_0(x)Z(x) = \hat{S}_M^T K_{\psi_o}^0 \Psi_M(x),\\
&\sum_{i=1}^{n} \varphi_i(x)\frac{\partial^i Y(x)}{\partial x^i} = \hat{R}_M^T \sum_{i=1}^{n} K_{\varphi_i}^i \Psi_M(x), && \sum_{i=1}^{n} \varrho_i(x)\frac{\partial^i Z(x)}{\partial x^i} = \hat{S}_M^T \sum_{i=1}^{n} K_{\varrho_i}^i \Psi_M(x),\\
&\varphi_0(x)Y(x) = \hat{R}_M^T K_{\varphi_o}^0 \Psi_M(x), && \varrho_0(x)Z(x) = \hat{S}_M^T K_{\varrho_o}^0 \Psi_M(x).
\end{aligned} \tag{4.14}$$

Approximating the source terms $f(x)$ and $g(x)$ with Jacobi polynomials along with using (4.14),(4.11) in (4.10) and writing in vector notation we get

$$\begin{aligned}\begin{bmatrix} R_M^T\Psi_M(x) \\ S_M^T\Psi_M(x)\end{bmatrix} &= \begin{bmatrix} \hat{R}_M^T\sum_{i=1}^n K_{\phi_i}^i\Psi_M(x) \\ \hat{S}_M^T\sum_{i=1}^n K_{\varrho_i}^i\Psi_M(x)\end{bmatrix} + \begin{bmatrix} \hat{S}_M^T\sum_{i=1}^n K_{\psi_i}^i\Psi_M(x) \\ \hat{R}_M^T\sum_{i=1}^n K_{\varphi_i}^i\Psi_M(x)\end{bmatrix} \\ &+ \begin{bmatrix} \hat{R}_M^T K_{\phi_0}^0\Psi_M(x) \\ \hat{S}_M^T K_{\varrho_0}^0\Psi_M(x)\end{bmatrix} + \begin{bmatrix} \hat{S}_M^T K_{\psi_0}^0\Psi_M(x) \\ \hat{R}_M^T K_{\varphi_0}^0\Psi_M(x)\end{bmatrix} + \begin{bmatrix} \hat{F}_M\Psi_M(x) \\ \hat{G}_M\Psi_M(x)\end{bmatrix}.\end{aligned} \tag{4.15}$$

With out loss of ambiguity we can drop the index of i and rewrite (4.15) as

$$\begin{bmatrix} R_M^T\Psi_M(x) \\ S_M^T\Psi_M(x)\end{bmatrix} = \begin{bmatrix} \hat{R}_M^T\sum_{i=0}^n K_{\phi_i}^i\Psi_M(x) \\ \hat{S}_M^T\sum_{i=0}^n K_{\varrho_i}^i\Psi_M(x)\end{bmatrix} + \begin{bmatrix} \hat{S}_M^T\sum_{i=0}^n K_{\psi_i}^i\Psi_M(x) \\ \hat{R}_M^T\sum_{i=0}^n K_{\varphi_i}^i\Psi_M(x)\end{bmatrix} + \begin{bmatrix} \hat{F}_M\Psi_M(x) \\ \hat{G}_M\Psi_M(x)\end{bmatrix}. \tag{4.16}$$

Taking the transpose of the (4.16) and after a short factorization we get

$$\begin{aligned}\begin{bmatrix} R_M^T & S_M^T\end{bmatrix} A &= \begin{bmatrix} \hat{R}_M^T & \hat{S}_M^T\end{bmatrix}\begin{bmatrix} \sum_{i=0}^n K_{\phi_i}^i & O_{M\times M} \\ O_{M\times M} & \sum_{i=0}^n K_{\varrho_i}^i\end{bmatrix} A \\ &+ \begin{bmatrix} \hat{R}_M^T & \hat{S}_M^T\end{bmatrix}\begin{bmatrix} O_{M\times M} & \sum_{i=0}^n K_{\varphi_i}^i \\ \sum_{i=0}^n K_{\psi_i}^i & O_{M\times M}\end{bmatrix} A + \begin{bmatrix} \hat{F}_M & \hat{G}_M\end{bmatrix} A.\end{aligned} \tag{4.17}$$

Where $A = \begin{bmatrix} \Psi_M(x) & O_M \\ O_M & \Psi_M(x)\end{bmatrix}$, O_M is zero vector of order M and $O_{M\times M}$ is zero matrix of order M. Canceling out the common terms and using value of $\hat{R}_M^T$ and $\hat{S}_M^T$ from (4.13) in (4.17) we get

$$\begin{aligned}&\begin{bmatrix} R_M^T & S_M^T\end{bmatrix} - \begin{bmatrix} R_M^T & S_M^T\end{bmatrix}\begin{bmatrix} H_{M\times M}^{\eta,\sigma}\sum_{i=0}^n K_{\phi_i}^i & H_{M\times M}^{\eta,\sigma}\sum_{i=0}^n K_{\varphi_i}^i \\ H_{M\times M}^{\eta,\sigma}\sum_{i=0}^n K_{\psi_i}^i & H_{M\times M}^{\eta,\sigma}\sum_{i=0}^n K_{\varrho_i}^i\end{bmatrix} \\ &- \begin{bmatrix} F_1^T & F_2^T\end{bmatrix}\begin{bmatrix} \sum_{i=0}^n K_{\phi_i}^i & \sum_{i=0}^n K_{\varphi_i}^i \\ \sum_{i=0}^n K_{\psi_i}^i & \sum_{i=0}^n K_{\varrho_i}^i\end{bmatrix} - \begin{bmatrix} \hat{F}_M & \hat{G}_M\end{bmatrix} = 0.\end{aligned} \tag{4.18}$$

Which is generalized Lyapunov type matrix equation whose solution will lead us to the approximate solution of problem.

5 Examples

We check the efficiency of the proposed techniques by solving four test problems.The first two problems consists of single FDEs while the last two problem is devoted to coupled system of FDEs

Example 5.1. *Consider the following fractional differential equation*

$$\frac{\partial^\sigma Y(x)}{\partial x^\sigma} = e^x\frac{\partial Y(x)}{\partial x} + \cos(x)Y(x) - \pi^2\sin(\pi x) - \sin(\pi x)\cos(x) - \pi e^x\cos(\pi x), \tag{5.1}$$

with initial condition $Y(0) = 0$ and $Y'(0) = \pi$. Where $1 < \sigma \le 2$, $x \in [0, 3]$. The exact solution of this problem at $\sigma = 2$ is $Y(x) = sin(\pi x)$. We approximate the solution of this problem with the new technique and observe high accuracy of the solution. We simulate the algorithm at different scale level and as shown in $Fig\,(1)(a)$, the approximation becomes equal to the exact solution as scale level increases.The amount of absolute error is less than 10^{-2} see $Fig\,(1)(c)$. It is the basic property of fractional differential equation that the result for fractional order differential equation approaches to the result of integer order differential as the order of derivative approaches from fractional to integer. We approximate the solution at fractional value of σ and observe that as $\sigma \to 2$ the solution approaches uniformly to the exact solution at $\sigma = 2$, see $Fig\,(1)(b)$. Note that in all figures $\overbrace{Y(x)_M}$ represents approximate solution at scale level M.

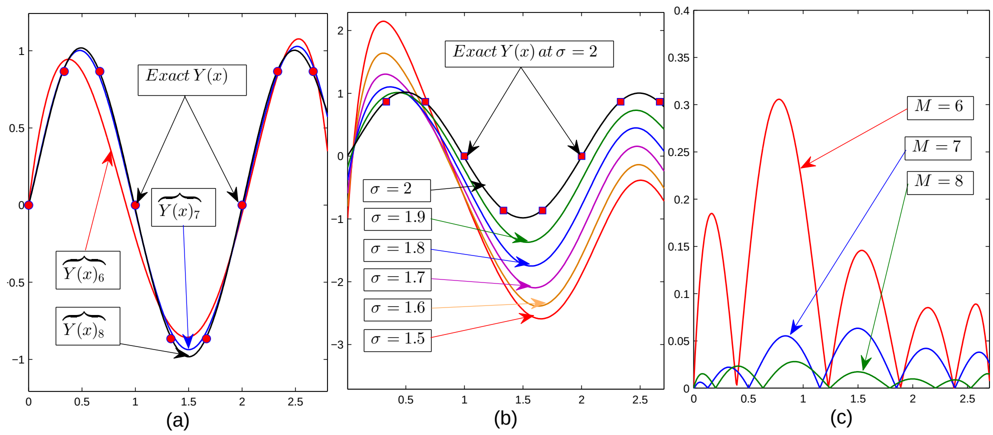

Fig. 1. (a)Approximate solution of Example 1 at different scale level. (b) Approximate solution of Example 1 at fractional value of σ and its comparison with the exact solution at $\sigma = 2$. (c) Absolute error at different scale level

Example 5.2. *Consider the following fractional differential equation*

$$\begin{aligned}\frac{\partial^\sigma W(x)}{\partial x^\sigma} =&(x^3-x^2+2)\frac{\partial W(x)}{\partial x}+(x^4+x^2-4)W(x)+\mathrm{e}^x(x^7+2\,x^5\\&-5\,x^4-x^3+3\,x^2-6\,x+2)\end{aligned} \tag{5.2}$$

with initial condition $W(0) = 0$ and $W'(0) = 0$, $1 < \sigma \leq 2$. One can easily check that the exact solution of the problem at $\sigma = 2$ is $W(x) = e^x(x^2 - x^3)$. We fix $\sigma = 2$ and approximate the solution at different scale level. We observe that as the scale level increase the approximation becomes equal to the exact solution. $Fig\,(2)(a)$ shows the comparison of exact solution with approximate solution at scale level $M = 5, 6$. We found that the absolute error is much more less than 10^{-3} at scale level $M = 6$ see $Fig\,(2)(c)$. We also the check the behavior of solution at fractional value of σ, and as expected the solution approaches to the exact solution as $\sigma \to 2$, see $Fig\,(2)(b)$.

Example 5.3. *Consider the following coupled system of fractional differential equation*

$$\begin{aligned}\frac{\partial^\sigma U(x)}{\partial x^\sigma} &= 2xU(x)+3x^2V(x)+cos(x)-3x^2cos(x)-2xsin(x)\\ \frac{\partial^\sigma V(x)}{\partial x^\sigma} &= 4x^3U(x)+x^2V(x)-sin(x)-x^2cos(x)-4x^3sin(x),\end{aligned} \tag{5.3}$$

where $0 < \sigma \leq 1$, $x \in [0, 3]$. The exact solution of this problem at $\sigma = 1$ is $U(x) = sin(x)$ and $V(x) = cos(x)$. We approximate the solutions of this problem with our new technique and found very high accuracy of the solution. We found that as the scale level increases the approximate solution becomes more and more accurate. And at scale level $M = 6$ the approximation becomes equal to the exact solutions see $Fig(3)(a)$. We observe that the error decreases significantly with the increase of scale level and at $M = 6$ the absolute error is much more less than 10^{-3}. $Fig\,(3)(b)$ and $Fig\,(3)(c)$ shows the absolute amount of error at different scale level in $U(x)$ and $V(x)$ respectively. We also approximate the solutions at fractional value of σ and we found that as $\sigma \to 1$ the approximate solution approaches the exact solution at $\sigma = 1$. $Fig\,(4)(a)$ and $Fig\,(4)(b)$ shows this phenomena for $U(x)$ and $V(x)$ respectively.

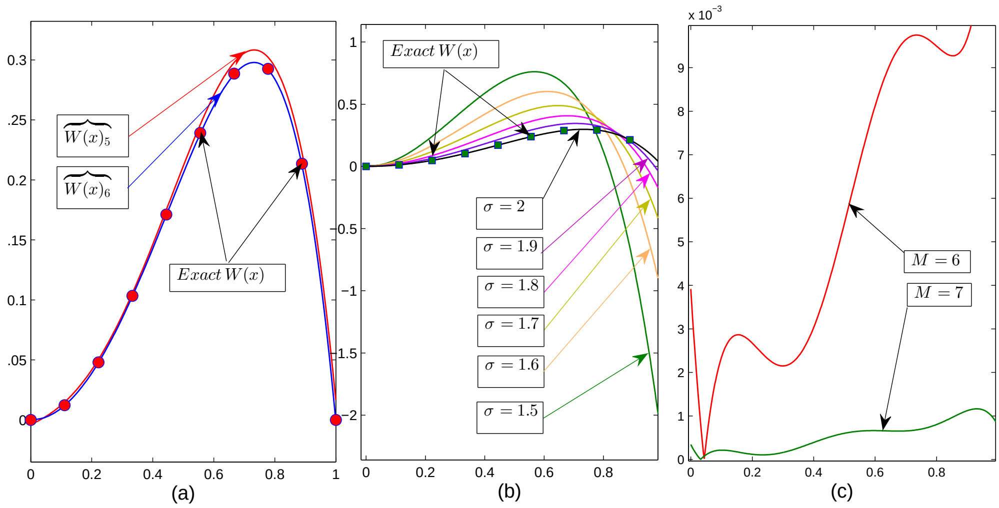

Fig. 2. (a) Comparison of exact and approximate solution of Example 2 at different scale level. (b) The approximate solution at fractional value of σ and its comparison with the exact solution at $\sigma = 2$. (c) Absolute error of Example 2 at different scale level

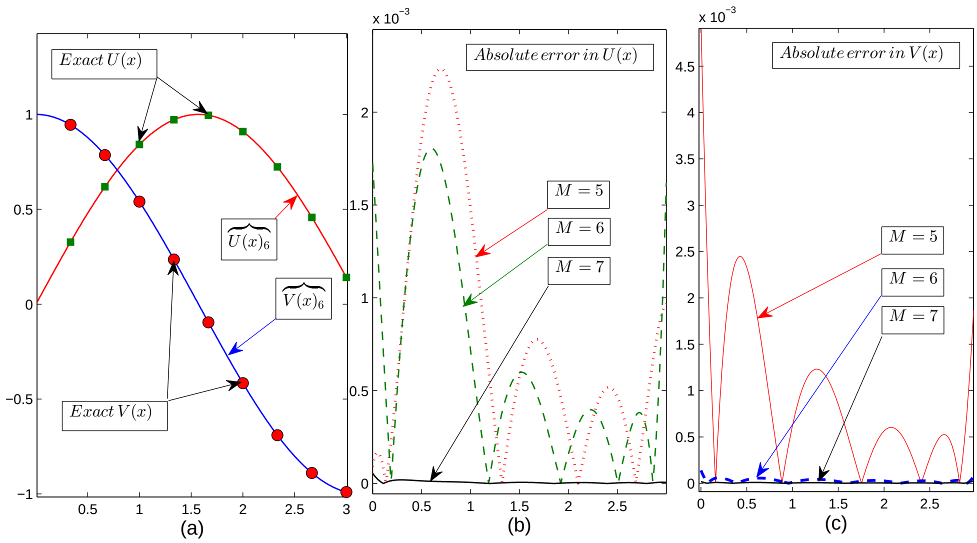

Fig. 3. a)Comparison of exact solutions of Example 3 with approximate solutions. (b) Absolute error in $U(x)$ at different scale level.(c) Absolute error in $V(x)$ at different scale level

Example 5.4. *Consider the following coupled system of FDEs*

$$\begin{aligned} \frac{\partial^{\sigma} U(x)}{\partial x^{\sigma}} &= 2x^2 \frac{\partial U(x)}{\partial x} + 3x^2 \frac{\partial V(x)}{\partial x} + f_1(x), \\ \frac{\partial^{\sigma} V(x)}{\partial x^{\sigma}} &= 4x^2 \frac{\partial U(x)}{\partial x} + x^3 \frac{\partial V(x)}{\partial x} + f_2(x), \end{aligned} \tag{5.4}$$

with initial condition

$$U(0) = U'(0) = V(0) = V'(0) = 0,$$

where the source terms are defined as $f_1(x) = x\left(-10\,x^6 + 8\,x^5 - 24\,x^4 + 9\,x^3 + 32\,x^2 - 12\,x + 12\right)$, *and* $f_2(x) = -24\,x^6 + 19\,x^5 - 20\,x^4 + 12\,x^2 - 6\,x - 4$.

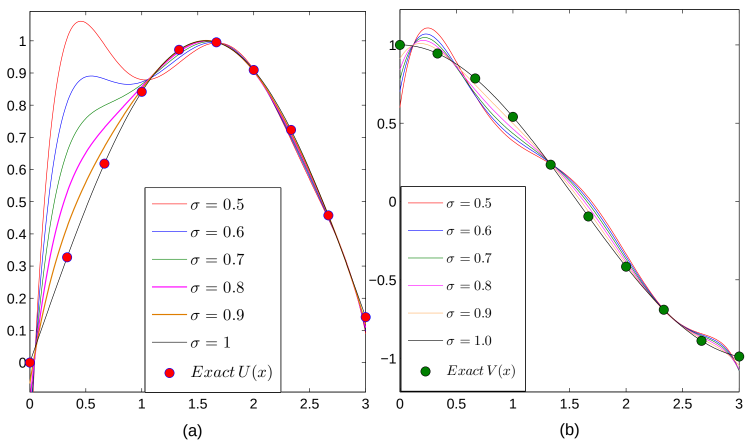

Fig. 4. (a)Approximate U(x) at fractional value of σ, ie $\sigma = 0.5 : .1 : 1$. (b)Approximate V(x) at fractional value of σ, ie $\sigma = 0.5 : .1 : 1$

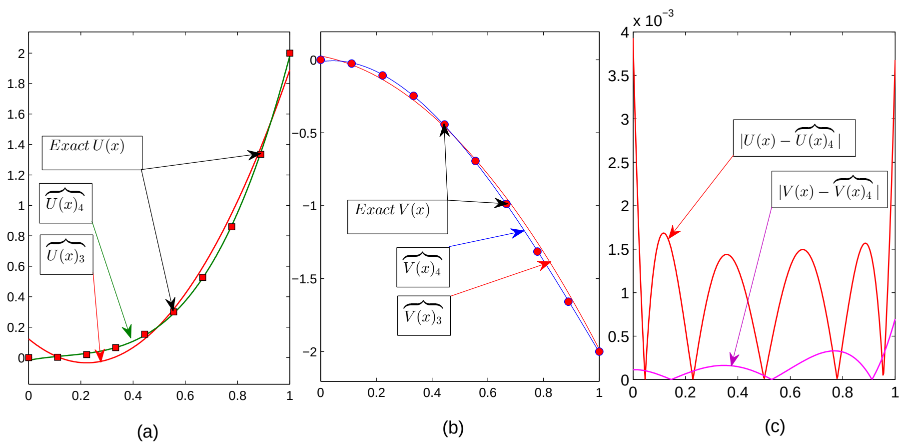

Fig. 5. (a)Comparison of exact and approximate U(x) of Example 4 at scale level $M = 3, 4$. (b)Comparison of exact and approximate V(x) of Example 4 at scale level $M = 3, 4$.(c) Absolute error in $U(x)$ and $V(x)$ at $M = 4$

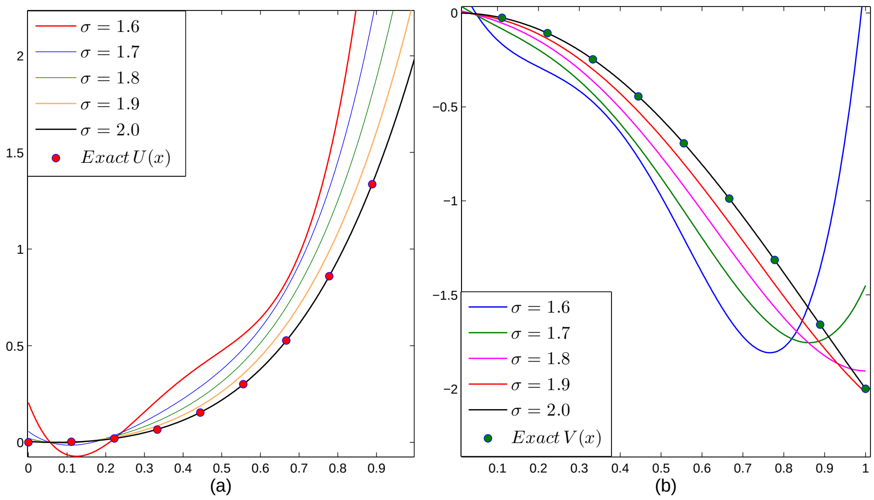

Fig. 6. (a) Approximate $U(x)$ of Example 4 at fractional value of σ. (b) Approximate $V(x)$ of Example 4 at fractional value of σ

Note that $1 < \sigma \leq 2$ and $x \in [0, 1]$. One can easily check that exact solution at $\sigma = 2$ is $U(x) = (x^5 - x^4) + 2x^3$ and $V(x) = (x^4 - x^3) - 2x^2$. We approximate the solution of this problem with our new technique and as expected we get a high accuracy of the solution. We observe that as the scale level increases the approximate solution becomes equal to the exact solution and scale level $M = 4$ the approximation is equal to the exact solutions. One can easily see this phenomena in Fig (5)(a) and Fig (5)(b). We also approximate the solution of the above problem at fractional value of σ and the same conclusion is made as the previous problem see Fig (6)(a) and Fig (6)(b). In Fig (5)(c) the absolute error at scale level $M = 4$ is shown for both solutions, One can easily see that the error is much more less than 10^{-3} even for very small scale level.

6 Conclusion

From the analysis and numerical experiments we concluded that the method is very efficient and provide a high accurate estimate of the solution. From the error analysis we see that the method can provide the error much more less than 10^{-3} for very small scale level. We believe that by increasing the scale level one can obtain more and more accurate solution.The method can be easily extended to solve such problems under different type of boundary conditions.It is also expected that the method can provide much more accurate solution by using some other kind of orthogonal polynomials like Laguerre,Berenstien etc.

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Competing Interests

The authors declare that no competing interests exist.

References

[1] Scalas E, Raberto M, Mainardi F. Fractional calclus and continous-time finance. Physics A, Sta. Mec. & App. 2000;284:376-384.

[2] Ross B. (Ed). Fractional calculus and its applications. Lecture Notes in Mathematics. Springer. 1975;457.

[3] Oldhalm KB, Spainer J. The fractional calculus. Academic Press, New York; 1974.

[4] Podlubny I. Fractional differential equations. Academic Press, New York; 1999.

[5] Manuel Duarte Origueira. Fractional calculus for scientists and engineers. Springer, Berlin; 2011.

[6] Rossikhin Yu, Shitikova MV. Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids. Appl. Mech. Rev. 1997;50;15-67.

[7] Concepcion A. Monje, Yangquan Chen, Blas M. Vinagre, Dingyu Xue, Vicente Feliu. Fractional-order systems and controls: Fundamentals and applications. Adva. Indus. Con., Springer-Verlag, London; 2010.

[8] Chen Y, Lv Z. Positive solutions for fractional differential equations with variable coefficients. Elec. J. Diff. Equ. 2012;1-10.

[9] Yildiray Keskin, Onur Karaoglu, Sema Servy, Galip Oturance. The approximate solution of high-order linear fractional differential equaitons with variable coefficients in term of generalised Taylor Polynomials. Mat. & Comp. App. 2011;16(3):617-629.

[10] Liu J, Xu L, Shen Y, Mao X, Li X. A method for solving variable coefficients initial-boundary equations. J. Com. & Inf. Sys. 2013;9(10):3801-3807.

[11] Hammad Khalil, Rahmat Ali Khan, Mohammed Al Smadi, Asad Freihat. Approximation of solution of time fractional order three-dimensional heat conduction problems with Jacobi Polynomials. Journal of Mathematics. 2015;47(1):35-56.

[12] Hammad Khalil, Rahmat Ali Khan, Numerical scheme for solution of coupled system of initial value fractional order Fredholm integro differential equations with smooth solutions. Journal of Mathematical Extension. 2015;9:39-58.

[13] Hammad Khalil, Rahmat Ali Khan. Extended spectral method for fractional order three-dimensional heat conduction problem. Progress in Fractional Differentiation and Applications. 2015;1(3):165-185.

[14] Hammad Khalil, Rahmat Ali Khan, Rashidi MM. Brenstien polynomials and its application to fractional differential equation. Computational method for fractional differential equations; 2015.

[15] Hammad Khalil, Rahmat Ali Khan, Mohammed Al-Samdi, Asad Friehat, Nabil Shawagfeh. New operational matrix for shifted Legendre polynomials and Fractional differential equations with variable coefficients. Journal of Mathematics, 2015;47(1).

[16] Hammad Khalil, Rahmat Ali Khan, Mohammed Al Samdi, Asad Friehat. A generalized algorithm based on Legendre polynomials for numerical solutions of coupled system of fractional order differential equations. Journal of Fractional Calculus and Applications. 2015;6(2):123-143.

[17] Kamal Shah, Hammad Khalil, Rahmat Ali Khan. Investigation to positive solution of fractional order impulsive differential equations. Chaos Solitons and Fractals. 2015;77:240-246.

[18] Irshad Khalil, Adnan Khalil, Sami Ur Rehman, Hammad Khalil, Rahmat Ali Khan. Classification of ECG signals using Legendre Moments. International Journal of Bioinformatics and Biomedical Engineering. 2015;1(3):284-291.

[19] Kamal Shah, Hammad Khalil, Rahmat Ali Khan. Upper and lower solution to coupled system of nonlinear fractional differetial equations. Progress in Fractional Differential Equations and Applications; 2015.

[20] Hammad khalil, Rahmat Ali Khan. The use of Jacobi Polynomials in the numerical solution of coupled system of fractional differential equations. Int. J. of Comp. Mat; 2014. Available:http://dx.doi.org/10.1080/00207160.2014.945919

[21] Saadatmandi A, Deghan M. A new operational matrix for solving fractional-order differential equation. Comp. & Mat. with App. 2010;59:1326-1336.

[22] Paraskevopolus PN, Saparis PD, Mouroutsos SG. The fourier series operational matrix of integration., Int. J. syst. sci. 1985;16:171-176.

[23] Doha EH, Bahraway AH, Ezz-Eliden SS. A Chebeshev spectral method based on operational matrix for initial and boundary value problems of fractional order. Com. & Mat. with App. 2011;62:2364-2373.

[24] Doha EH, Bahraway AH, Ezz-Eliden SS. Efficient Chebeshev spectral method for solving multi term fractional order differential equations, App. Mat. Mod. 2011;35:5662-5672.

[25] Doha EH, Bahraway AH, Ezz-Eliden SS. A new Jacobi operational matrix: An application for solving fractional differential equation. App. Mat. Mod. 2012;36:4931-4943.

[26] Razzaghi M, Yousefi S. Sine-cosine wavelets operational matrix of integration and its application in the calculas of variation. Int. J. Syst. Sci. 2002;33:805-810.

[27] Razzaghi M, Yousefi S. The legender wavelets operational matrix of integration. Int. J. Syst. Sci. 2001;32:495-502.

[28] Khalil H, khan RA. A new method based on legender polynomials for solution of system of fractional order partial differential equation. Int. J. Com. Mat. 2014;(ID: 880781). DOI: 10.1080/00207160.2014.880781

[29] Khalil H, Khan RA. New operational matrix of integration and coupled system of fredholm integral equations. Chinese J. of Mat; 2014.

[30] Khalil H, Khan RA. A new method based on Legendre polynomials for solutions of the fractional two-dimensional heat conduction equation, Comp. & Mat. with App. 2014;67:1938-1953.

[31] Bhrawy AH, Al-Shomrani MM. A shifted Legendre spectral method for fractional-order multi-point boundary value problems. Advances in Difference Equations. Springer; 2012.

[32] Bhrawy AH, Alghamdi MA. A shifted Jacobi-Gauss-Lobatto collocation method for solving nonlinear fractional Langevin equation involving two fractional orders in different interval. Boundary Value Problems. Springer; 2012.

[33] Tian WY, Deng WH, Wu YJ. Polynomial spectral collection method for the space fractional advection-diffusion equation. Numerical Methods for Partial Differential Equation.

[34] Esmaeili S, Shamsi M. A pseudo-spectral scheme for the approximate solution of a family of fractional differential equations. Comm. in Nonl. Sci. & Num. Sim. 2011;16:3646-3654.

[35] Li C, Zeng F, Liu F. Spectral approximation of the fracitonal integral and derivative. Frac. Cal. & App. Anal. 2014;15:241-258.

[36] Mishra VN. Some problems on approximations of functions in Banach spaces, Ph.D. Thesis. Indian Institute of Technology, Roorkee-247 667, Uttarakhand, India; 2007.

[37] Mishra VN, Mishra LN. Trigonometric approximation of signals (Functions) in Lp ($p \geq 1$) norm. International Journal of Contemporary Mathematical Sciences. 2012;7(19):909-918.

[38] Mishra VN, Khatri K,. Mishra LN. Deepmala, Inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators. Journal of Inequalities and Applications. 2013;2013:586.
DOI: 10.1186/1029-242X-2013-586.

[39] Deepmala. A study on fixed point theorems for nonlinear contractions and its applications, Ph.D. Thesis. Pt. Ravishankar Shukla University, Raipur (Chhatisgarh) India. 2014;492 010.

[40] Podlubny I. Fractional differential equations. Academic Press, San Diedo; 1999.

[41] Kilbas AA, Srivastava HM, Trujillo J. Theory and applications of fractional differential equations. Elsevier Science, Amsterdam; 2006.

[42] Gasea M, Sauer T. On the history of multivariate polynomial interpolation. J. Comp. & App. Mat. 2000.122:23-35.

[43] Nemati S, Ordokhani Y. Legeder expansion methods for the numerical solution of nonlinear 2D Fredholm integral equations of the second kind. J. Appl. Math. and Infor. 2013;31:609-621.

[44] Kalateh Bojdi Z, Ahmadi-Asl S, Aminataei A. The general two dimensional shifted Jacobi matrix method for solving the second order linear partial difference-differential equations with variable coefficients. Uni. J. App. Mat. 2013;1(2):142-155.

[45] Hesthaven JS, Gottlieb S, Gottlieb D. Spectral methods for time-dependent problems. Ist ed, Cambridge University; 2007.

[46] Doha EH, Bhrawy AH, Ezz-Eldien SS. A new Jacobi operational matrix: An application for solving fractional differential equations. Appl. Math. Modell. 2012;36:4931-4943.

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