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A New Class of Ordinary Differential Equation at Infinity and its Solution

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Abstract

The main purpose of this paper is to construct a new class of second order differential equation at infinity. To solve the problem, we generate the auxiliary equation via a pre-auxiliary equation and obtain the general solution of it. We express the higher order of the differential equation in a matrix form. We have also studied the problem with a change in variable. We prove the sufficient condition of the solutions for the 2nd order differential equation under certain conditions for existence of the problem.

Keywords: Ordinary differential equation, pre-auxiliary equation; auxiliary equation.

2010 Mathematics Subject Classification: 34A-05;

1 Introduction

Studying of the behavior of the physical, engineering and other problems (modeling), the theory of ordinary differential equation gives the standard methods. The researchers have done many

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interesting results on this field of work. For our reference we recall the works of the researchers, such as Rubinstein [1], Kreyszig [2], Agarwal and ÓRegan [3], Boyce and DiPrima [4], to name only a few. Boyce and DiPrima [4] have studied the stability of the problem (1.2) by making the change of variable $\xi = 1/x$ and studying the resulting equation at $\xi = 0$.

The general form of ordinary differential equation (ODE) of n^{th} order is

$$p_0(x)y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-2}(x)y'' + p_{n-1}(x)y' + p_n(x)y = 0, \tag{1.1}$$

where p_i's are arbitrary functions of x.

An ordinary differential equation called the Sturm-Liouville equation of the form

$$y'' + P(x)y' + Q(x)y = 0 \tag{1.2}$$

has singularities for finite $x = x_0$ under the following conditions:

(a) If either $P(x)$ or $Q(x)$ diverges as $x \to x_0$, $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ remain finite as $x \to x_0$, then x_0 is the regular or non-essential singular point,

(b) if $P(x)$ diverges faster than $(x-x_0)^{-1}$ so that $(x-x_0)P(x) \to \infty$ as $x \to x_0$ or $Q(x)$ diverges faster than $(x - x_0)^{-2}$ so that $(x - x_0)^2Q(x) \to \infty$ as $x \to x_0$, then x_0 is the the irregular or essential singular point

Singularities of (1.2) are investigated by Morse and Feshbach [5]. They have substituted $x = z^{-1}$ in the above equation to get the reduced equation

$$z^4\frac{d^2y}{dz^2} + \left(2z^3 - z^2P\left(z^{-1}\right)\right)\frac{dy}{dz} + Q\left(z^{-1}\right)y = 0$$

and investigated its singularities at infinity as follows:

(a) if

$$\alpha(z) \equiv \frac{2z - P\left(z^{-1}\right)}{z^2} \quad \text{and} \quad \beta(z) \equiv \frac{Q\left(z^{-1}\right)}{z^4}$$

remain finite at $x = \pm\infty$ (i.e., $z = 0$), then the point is ordinary,

(b) if $\alpha(z)$ diverges no more rapidly than $\frac{1}{z}$ and $\beta(z)$ diverges no more rapidly than $\frac{1}{z^2}$, then the point is a regular or non-essential singular point, otherwise the point is a irregular or essential singular point.

Morse and Feshbach ([5], 1953, pp. 667-674) has given the canonical forms and solutions for second-order ordinary differential equations classified by types of singular points. Došlá and Kiguradze [6] have studied the solution of a new class of second order differential equation on the dimension of spaces of vanishing at infinity. They have also studied the auxiliary equation of it. Makino [7] has studied the existence of positive solutions at infinity for ordinary differential equations of Emden Type. To extend the results of Philos et al. [8], Philos ad Tsamatos [9] have studied the differential equation of n-th order $(n > 1)$ nonlinear ordinary differential equation

$$y^{(n)}(x) = f(x, y(x), y'(x), \cdots, y^{(N)}(x)), x \geq x_0 > 0, \tag{$*$}$$

where N is an integer with $0 \leq N \leq n - 1$, and f is a continuous real-valued function on $[x_0, \infty) \times R^{N+1}$. They studied the condition of all solutions to be Asymptotic to Polynomials at Infinity.

In this paper, we introduce a special type of homogeneous differential equation at infinity having upper bounded functions. We introduce a class of ordinary differential equation defined by

$$z^4\frac{d^2y}{dz^2} + (2z^3 - az^2)\frac{dy}{dz} + by = 0, \tag{DE$_\infty$}$$

with conditions

$$\lim_{x\to 0} z(x) = 1 \quad \text{and} \quad \lim_{x\to\infty} z(x) = 0$$

where a and b are constants. Here $z(x) \to 0$ as $x \to \infty$ more faster than $\frac{1}{x}$, we call the above differential equation as differential equation at infinity (in short; DE_∞). We study the existence of the solution of (DE_∞) via a pre-auxiliary equation. We have also studied the alternative method to solve like a ODE.

2 Second Order Differential Equation and Its Solution

For our need, we make the following definition.

Definition 2.1. *The equation* (1.1) *is an entire homogeneous differential equation if the coefficients* $y, y', y'', \cdots, y^{(n)}$ *are all bounded and entire functions such that*

$$\frac{p_i(x)}{p_0(x)}, i = 0, 1, \cdots, n-1$$

are all bounded and entire functions where the primes are number of derivative of y *with respect to* x.

2.1 Generalized Homogeneous Logarithmic Differential Equation

In this section, we study the concept of generalized homogeneous differential equation of second order with exponential solution.

Consider the second order initial value problem having generalized homogeneous logarithmic differential equation of the form

$$e^{-2x}y'' + \phi(e^{-x}, e^{-2x})y' + by = 0 \tag{2.1}$$

with initial conditions

$$y(x_0) = y_0 \quad \text{and} \quad y'(x_0) = z_0$$

where b is a real constant and the number of primes ($'$) are the order of derivatives of the entire function y with respect to x and $\phi(e^{-x}, e^{-2x})$ is the function of linear combination of e^{-x} and e^{-2x} given by

$$\phi(e^{-x}, e^{-2x}) = a_1e^{-x} + a_2e^{-2x},$$

with $a_2 \neq 0$ and a_2 is independent on a_1. Now

$$\lim_{x\to 0} \phi(e^{-x}, e^{-2x}) = a_1 + a_2$$

exists for finite a_1 and a_2. Assume that

$$y = \exp(\lambda e^x)$$

be the solution of (2.1), then we have

$$\begin{aligned} y' &= \lambda y e^x, \\ y'' &= \lambda(\lambda e^{2x} + e^x)y \\ y''' &= \lambda(\lambda^2 e^{3x} + 3\lambda e^{2x} + e^x)y \\ &\vdots \end{aligned}$$

Putting the values y' and y'' in (2.1), the equation obtained is

$$\left[\lambda(\lambda + e^{-x}) + \lambda\phi(e^{-x}, e^{-2x})e^x + b\right] y = 0.$$

For nontriviality of y, we have

$$\lambda(\lambda + e^{-x}) + \lambda(a_1 + a_2 e^{-x}) + b = 0. \tag{2.2}$$

Since a_2 is independent on a_1, the above equation can be written as

$$\lambda^2 + a_1\lambda + (a_2 + 1)\lambda e^{-x} + b = 0, \tag{2.3}$$

For simplicity, we call the above equation as *pre-auxiliary* (or *pre-indicial*) equation. Since λ is arbitrary, so to remove the term e^{-x} from (2.3), we have to take coefficient of e^{-x} to zero which gives $a_2 = -1$. Taking limit as $x \to 0$ in the pre-auxiliary equation (2.3) and putting the value of a_2, we get the auxiliary (or indicial) equation of (2.1) is

$$\lambda^2 + a_1\lambda + b = 0 \tag{2.4}$$

having the roots $\lambda = \lambda_1, \lambda_2$. Hence (2.1) reduces to

$$e^{-2x}y'' + \left(a_1 e^{-x} - e^{-2x}\right) y' + by = 0,$$

whose solution is

$$y(x) = \begin{cases} c_1 \exp(\lambda_1 e^x) + c_2 \exp(\lambda_2 e^x) & \text{if } \lambda_1 \neq \lambda_2, \\ (c_1 + c_2 e^x) \exp(\lambda e^x) & \text{if } \lambda = \lambda_1 = \lambda_2, \\ \exp(\alpha e^x) \left[A \cos(\beta e^x) + B \sin(\beta e^x)\right] & \text{if } \lambda = \alpha \pm i\beta. \end{cases}$$

Remark 2.1. *The equation* (2.2) *can be written as*

$$\lambda \left(\lambda + \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} \right) + \lambda \left(a_1 + a_2 \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} \right) + b = 0, \tag{2.5}$$

giving

$$\lambda^2 + a_1\lambda + b = 0$$

and

$$(a_2 + 1)\lambda + \mathcal{O}(x) = 0$$

satisfying $\lim_{x\to 0} \mathcal{O}(x) = 0$. *Taking limit as* $x \to 0$ *in the above equation and equating the coefficients of* λ *to zero, we get* $a_2 = -1$.

Remark 2.2. *The function* $y = \exp(\lambda e^x)$ *solves* (2.1) *if only if* $a_2 = -1$.

2.2 Differential Equation at Infinity

Let $z = z(x)$, $x > 0$ be a bounded entire function. We define a class of ordinary differential equation at infinity defined by

$$z^4 \frac{d^2y}{dz^2} + (2z^3 - az^2)\frac{dy}{dz} + by = 0, \tag{2.6}$$

with conditions

$$\lim_{x\to 0} z(x) = 1 \quad \text{and} \quad \lim_{x\to\infty} z(x) = 0$$

where a and b are constants. Since $e^{-x} \to 0$ as $x \to \infty$ and $e^{-x} \to 1$ as $x \to 0$, taking $z = e^{-x}$, we get

$$\begin{aligned} \frac{dy}{dz} &= -\frac{1}{z}y' \\ \frac{d^2y}{dz^2} &= \frac{1}{z^2}(y' + y'') \end{aligned}$$

Putting the value of $\frac{dy}{dz}$ and $\frac{d^2y}{dz^2}$ in the differential equation at infinity (2.6), we get

$$e^{-2x}y'' + \left(ae^{-x} - e^{-2x}\right)y' + by = 0 \tag{2.7}$$

which is a generalized homogeneous logarithmic differential equation (2.1) where

$$\phi(e^{-x}, e^{-2x}) = ae^{-x} - e^{-2x},$$

i.e., $a_1 = a$ and $a_2 = -1$. Thus the auxiliary equation obtained using (2.4) is

$$\lambda^2 + a\lambda + b = 0. \tag{2.8}$$

In particular, for any real positive n if $a = 0$ and $b = n^2$, then (2.6) reduces to

$$z^4\frac{d^2y}{dz^2} + 2z^3\frac{dy}{dz} + n^2y = 0, z = z(x); \tag{2.9}$$

with conditions

$$\lim_{x\to 0} z(x) = 1 \quad \text{and} \quad \lim_{x\to\infty} z(x) = 0$$

has the bases $\cos(ne^x)$ and $\sin(ne^x)$. For $n = 1$, the bases $\cos(e^x)$ and $\sin(e^x)$ of (2.9) are shown in Figure-1 which are plotted using Matlab taking $x \in [-10, 10]$.

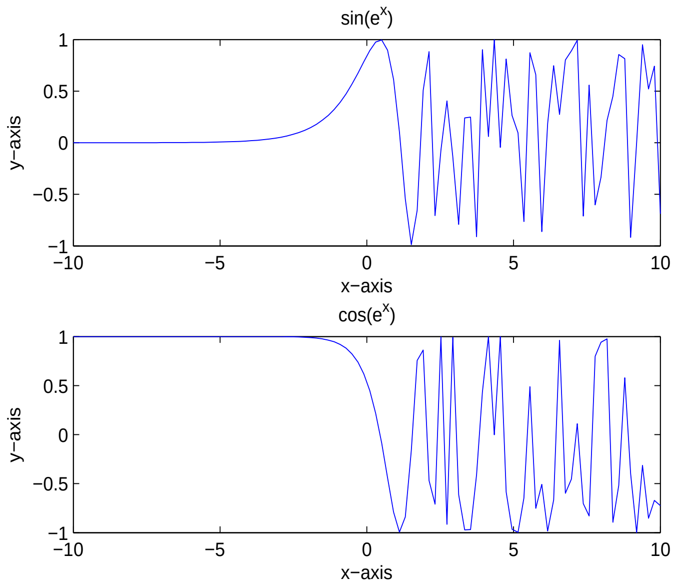


Figure 1: Basic functions

Note 2.1. If $b = b_1 + ib_2$ with $|b| = 1$ and $|x| < l$, then the general solution y of the differential equation (2.9) moving in the disk of radius $R = \exp(b_1u - b_2v)$ where

$$u = e^{\operatorname{Re} x}\cos(\operatorname{Im} x) \quad \text{and} \quad v = e^{\operatorname{Re} x}\sin(\operatorname{Im} x),$$

$\operatorname{Re} x$ denotes the real part of x and $\operatorname{Im} x$ denotes the imaginary part of x.

Example 2.2. If $a = 3$ and $b = 2$, then from (2.8), we get $\lambda = -1, -2$. Thus the basis are $y_1 = \exp(-e^x)$ and $y_1 = \exp(-2e^x)$. Since $y = Ay_1 + By_2$ solves the differential equation

$$e^{-2x}y'' + \left(3e^{-x} - e^{-2x}\right) y' + 2y = 0,$$

the function $y = A\exp\left(-\frac{1}{z}\right) + B\exp\left(-\frac{2}{z}\right)$ solves the differential equation

$$z^4\ddot{y} + \left(2z^3 - 3z^2\right)\dot{y} + 2y = 0$$

at infinity where $z = e^{-x}$ and $\dot{y}$ denotes the derivative of y with respect to z.

3 Alternative Method: Logarithmic Conversion

We have studied the alternative method to find the auxiliary equation of the homogeneous logarithmic differential equation (2.7) of the form

$$e^{-2x}y'' + \left(ae^{-x} - e^{-2x}\right) y' + by = 0$$

where a, b are real constants.

Taking $x = \ln t$, $t \neq 0$, we get

$$\begin{aligned} y' &= t\dot{y} \\ y'' &= t\dot{y} + t^2\ddot{y} \end{aligned}$$

where the number of dots represents the order of derivatives of y with respect to t. From (2.7), we get

$$t^{-2}(t\dot{y} + t^2\ddot{y}) + \left(at^{-1} - t^{-2}\right) t\dot{y} + by = 0,$$
$$\ddot{y} + a\dot{y} + by = 0$$

which is same as auxiliary equation (2.8). Hence the solution of the differential equation obtained is

$$y(t) = \begin{cases} c_1\exp(\lambda_1 t) + c_2\exp(\lambda_2 t) & \text{if } \lambda_1 \neq \lambda_2, \\ (c_1 + c_2 t)\exp(\lambda t) & \text{if } \lambda = \lambda_1 = \lambda_2, \\ \exp(\alpha t)\left[A\cos(\beta t) + B\sin(\beta t)\right] & \text{if } \lambda = \alpha \pm i\beta. \end{cases}$$

Replacing t by e^x, we have

$$y(x) = \begin{cases} c_1\exp(\lambda_1 e^x) + c_2\exp(\lambda_2 e^x) & \text{if } \lambda_1 \neq \lambda_2, \\ (c_1 + c_2 e^x)\exp(\lambda e^x) & \text{if } \lambda = \lambda_1 = \lambda_2, \\ \exp(\alpha e^x)\left[A\cos(\beta e^x) + B\sin(\beta e^x)\right] & \text{if } \lambda = \alpha \pm i\beta. \end{cases}$$

Remark 3.1. *In particular, if $a = b = 0$, then* (2.7) *coincides with the ordinary differential equation*

$$y'' - y' = 0$$

having the solution

$$y(x) = c_1 + c_2e^x.$$

Example 3.1. The equation

$$e^{-2x}y'' + (-2e^{-x} - e^{-2x})y' + y = 0.$$

has the auxiliary equation

$$\lambda^2 - 2\lambda + 1 = 0$$

gives $\lambda = 1, 1$. Hence the solution is

$$y(x) = (c_1 + c_2e^x)\exp(e^x).$$

Example 3.2. The equation

$$e^{-2x}y'' + (e^{-x} - e^{-2x})y' - 2y = 0.$$

has the auxiliary equation

$$\lambda^2 + \lambda - 2 = 0$$

gives $\lambda = 1, -2$. Hence the solution is

$$y(x) = c_1 \exp(e^x) + c_2 \exp(-2e^x).$$

Example 3.3. The equation

$$e^{-2x}y'' + (-4e^{-x} - e^{-2x})y' + 13y = 0.$$

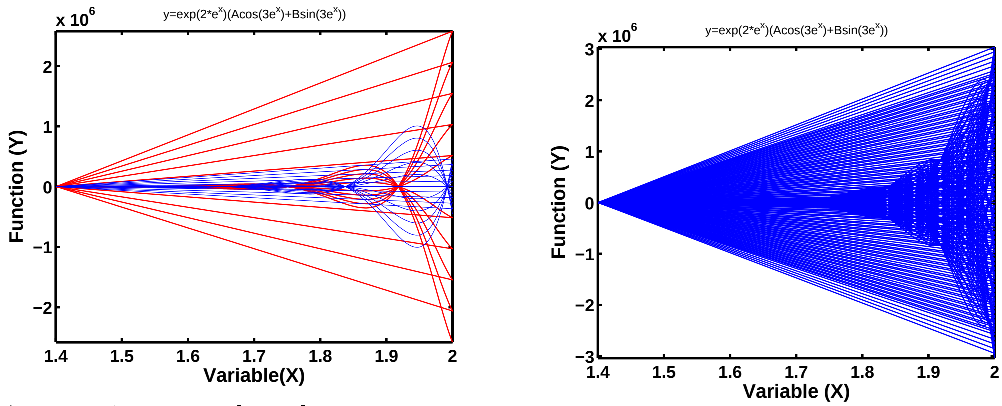
has the auxiliary equation

$$\lambda^2 - 4\lambda + 13 = 0$$

gives $\lambda = 2 \pm 3i$. Hence the solution is

$$y(x) = \exp(2e^x)\,(A\cos(3e^x) + B\sin(3e^x)) \tag{3.1}$$

The trajectories of $y(x)$ given in (3.1) is shown in Figure 2(a) and Figure 2(b) using MATLAB. The value $y(0) = 0$ exists for $A = 0$ and $B = 0$. In Figure 2(a), for blue figure has the value of $A = 0$ and $B \in [-1, 1]$ with space difference 0.2; for red figure has the value of $B = 0$ and $A \in [-1, 1]$ with space difference $h = 0.2$ where $x \in [1.4, 2]$ with space difference $h = 0.002$. In Figure 2(b), the values of A and B lie in the interval [-1,1] with space difference 0.2 where $x \in [1.4, 2]$ with space difference 0.002.



(a) Blue: $A = 0, B \in [-1, 1]$ and Red: $B = 0, A \in [-1, 1]$

(b) $A \in [-1, 1], \quad B \in [-1, 1]$

Figure 2: Sleeping Tower

4 Higher Order Homogeneous Logarithmic Differential Equations and Its Matrix form

The second order homogeneous logarithmic differential equation (2.7) can be written as a matrix form given by

$$PY = 0$$

where

$$P = \begin{pmatrix} p_0 & 0 & 0 \\ 0 & p_1 & 0 \\ 0 & 0 & p_2 \end{pmatrix} = \begin{pmatrix} e^{-2x} & 0 & 0 \\ e^{-2x} & e^{-x} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}$$

and

$$Y = \begin{pmatrix} y'' & 0 & 0 \\ 0 & y' & 0 \\ 0 & 0 & y \end{pmatrix} = \begin{pmatrix} D^2 & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} y & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & y \end{pmatrix}$$

where $D = \frac{d}{dx}$ as

$$\begin{aligned} p_0 &= e^{-2x}, \\ p_1 &= -e^{-2x} + ae^{-x}, \\ p_2 &= b \end{aligned}$$

satisfying the conditions in p_i's are

$$\sum \text{coeff}(e^{-2x}) = 0,$$
$$\sum \text{coeff}(e^{-x}) = a,$$
$$\sum \text{Constants} = b.$$

The general form of the third order homogeneous differential equation is

$$e^{-3x}y''' + (ae^{-2x} - 3e^{-3x})y'' + (be^{-x} - ae^{-2x} + 2e^{-3x})y' + cy = 0 \tag{4.1}$$

where a, b and c are real constants. The equation (4.1) can be written in matrix form given by

$$PY = 0$$

where

$$P = \begin{pmatrix} p_0 & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{pmatrix}$$

and

$$Y = \begin{pmatrix} y''' & 0 & 0 & 0 \\ 0 & y'' & 0 & 0 \\ 0 & 0 & y' & 0 \\ 0 & 0 & 0 & y \end{pmatrix}$$

with

$$\begin{aligned} p_0 &= e^{-3x} \\ p_1 &= -3e^{-3x} + ae^{-2x}, \\ p_2 &= 2e^{-3x} - ae^{-2x} + be^{-x}, \\ p_3 &= c, \end{aligned}$$

satisfying the conditions in p_i's are

$$\sum \text{coeff}(e^{-3x}) = 0,$$
$$\sum \text{coeff}(e^{-2x}) = 0,$$
$$\sum \text{coeff}(e^{-x}) = b,$$
$$\sum \text{Constants} = c.$$

Taking $x = \ln t$, $t > 0$ in (4.1), we get the reduced equation

$$\dddot{y} + a\ddot{y} + b\dot{y} + cy = 0$$

which is solvable.

The general form of the fourth order homogeneous logarithmic differential equation is

$$\begin{aligned} & e^{-4x}y^{iv} + (\alpha e^{-3x} - 6e^{-4x})y''' + (\beta e^{-2x} - 3\alpha e^{-3x} + 11e^{-4x})y'' \\ & + (\gamma e^{-x} - \beta e^{-2x} + 2e^{-3x} - 6e^{-4x})y' + \delta y = 0 \end{aligned} \tag{4.2}$$

where α, β, γ and δ are real constants. The equation (4.2) can be written in matrix form given by

$$PY = 0$$

where

$$P = \begin{pmatrix} p_0 & 0 & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 & 0 \\ 0 & 0 & p_3 & 0 & 0 \\ 0 & 0 & 0 & p_4 & 0 \\ 0 & 0 & 0 & 0 & p_5 \end{pmatrix}$$

and

$$Y = \begin{pmatrix} y^{iv} & 0 & 0 & 0 & 0 \\ 0 & y''' & 0 & 0 & 0 \\ 0 & 0 & y'' & 0 & 0 \\ 0 & 0 & 0 & y' & 0 \\ 0 & 0 & 0 & 0 & y \end{pmatrix}$$

with

$$\begin{aligned} p_0 &= e^{-4x}, \\ p_1 &= -6e^{-4x} + \alpha e^{-3x}, \\ p_2 &= 11e^{-4x} - 3\alpha e^{-3x} + \beta e^{-2x}, \\ p_3 &= -6e^{-4x} + 2\alpha e^{-3x} - \beta e^{-2x} + \gamma e^{-x}, \\ p_4 &= \delta \end{aligned}$$

satisfying the conditions in p_i's are

$$\begin{aligned} &\sum \text{coeff}(e^{-4x}) = 0, \\ &\sum \text{coeff}(e^{-3x}) = 0, \\ &\sum \text{coeff}(e^{-2x}) = 0, \\ &\sum \text{coeff}(e^{-x}) = \gamma, \\ &\sum \text{Constants} = \delta. \end{aligned}$$

Taking $x = \ln t$, $t > 0$ in (4.2), we get the reduced equation

$$\ddddot{y} + \alpha\dddot{y} + \beta\ddot{y} + \gamma\dot{y} + \delta dy = 0$$

which is solvable.

4.1 Formation of the Logarithmic Homogeneous Differential Equation

Let the transpose of X be denoted by X^{T}. Let the derivative of y with respect to x and t be denoted by $y^{(n)}$ and $\dot{y}^{(n)}$ respectively. The *logarithmic homogeneous differential equation* of n^{th} order is of form

$$A_E Y_x = 0 \tag{4.3}$$

where $A_E = A \cdot E$, $A = (a_{ij})$ being the n^{th} order matrix having the elements given by

$$(a_{ij}) = \begin{cases} a_{ii} = 1, & \text{for } i = 1; \\ a_{ij} \ncong 0, & \text{for } i \geq j; \\ a_{ij} = 0, & \text{for } i < j; \\ a_{nj} = 0, & \text{for } 1 \leq j < n; \end{cases}$$

$$E = \begin{pmatrix} e^{-nx} & e^{-(n-1)x} & \cdots & e^{-2x} & e^{-x} & 1 \end{pmatrix}^{\mathrm{T}}$$

and

$$Y_x = \begin{pmatrix} y^{(n)} & y^{(n-1)} & \cdots & y'' & y' & y \end{pmatrix}^{\mathrm{T}}.$$

Here the value $a_{ij} \ncong 0$ means all a_{ij}'s are not zero. Let A be decomposed by the rule $A = D(A) + C$ where $D(A)$ is a diagonal matrix containing diagonal elements of A and $C = (c_{ij})$ is the square matrix having the elements defined by

$$(c_{ij}) = \begin{cases} c_{i1} = 0, & \text{for } 1 \leq i \leq n; \\ c_{1j} = 0, & \text{for } 1 \leq j \leq n; \\ c_{ij} = k_{ij}, & \text{for } i \neq 1, i \geq j \geq 2, j \neq n; \\ c_{ij} = 0, & \text{for } i < j; \\ c_{in} = 0, & \text{for } 1 \leq i \leq n; \\ c_{nj} = 0, & \text{for } 1 \leq j \leq n, \end{cases}$$

where k_{ij}'s are the affine function of a_{ij}'s.

The equation (4.3) can be converted to

$$D_T Y_t = 0 \tag{4.4}$$

where $D_T = D \cdot T$,

$$T = \begin{pmatrix} 1 & t^{-1} & t^{-2} & \cdots & t^{-(n-1)} & t^{-n} \end{pmatrix}^{\mathrm{T}}$$

and

$$Y_t = \begin{pmatrix} \dot{y}^{(n)} & \dot{y}^{(n-1)} & \cdots & \dot{y}'' & \dot{y}' & y \end{pmatrix}^{\mathrm{T}}$$

by considering $x = \ln t$, $t > 0$. The equation (4.4) is solvable like a ordinary differential equation of n^{th} order if D_T has the decomposition of the form

$$D_T = D(A) + C_T$$

with C_T the matrix satisfies the condition $C_T = C \cdot T \equiv 0$. The value of a_{ij}'s can be obtained by solving equations

$$\text{coefficient of } t^{-m} = 0,\ m = 1, 2, \cdots, n$$

obtained from the equation $C_T Y_t = 0$

We have shown the formation of the logarithmic homogeneous differential equation of 4^{th} order. The logarithmic homogeneous differential equation of 4^{th} order is of form

$$A_E Y_x = 0$$

where $A_E = A \cdot E$,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ a_1 & \alpha & 0 & 0 & 0 \\ b_2 & b_1 & \beta & 0 & 0 \\ c_3 & c_2 & c_1 & \gamma & 0 \\ 0 & 0 & 0 & 0 & \delta \end{bmatrix},$$

$$E = \begin{pmatrix} e^{-4x} & e^{-3x} & e^{-2x} & e^{-x} & 1 \end{pmatrix}^{\mathrm{T}}$$

and

$$Y_x = \begin{pmatrix} y^{iv} & y''' & y'' & y' & y \end{pmatrix}^{\mathrm{T}}.$$

Taking $x = \ln t$, $t > 0$, the reduced equation obtained is

$$D_T Y_t = 0 \tag{4.5}$$

where $D_T = D \cdot T$,

$$T = \begin{pmatrix} 1 & t^{-1} & t^{-2} & t^{-3} & t^{-4} \end{pmatrix}^{\mathrm{T}}$$

and

$$Y_t = \begin{pmatrix} \dot{y}^{iv} & \dot{y}''' & \dot{y}'' & \dot{y}' & y \end{pmatrix}^{\mathrm{T}}.$$

Decomposing the matrix D_T by the rule given above we get

$$D_T = D(A) + C_T = D(A) + C \cdot T$$

where

$$D(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & \delta \end{bmatrix},$$

and

$$C_T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & c_{22} & 0 & 0 & 0 \\ 0 & c_{32} & c_{33} & 0 & 0 \\ 0 & c_{42} & c_{43} & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ t^{-1} \\ t^{-2} \\ t^{-3} \\ t^{-4} \end{pmatrix}$$

having elements

$$\begin{aligned} c_{22} &= a_1 + 6 \\ c_{32} &= b_1 + 3\alpha \\ c_{33} &= 3a_1 + b_2 + 7 \\ c_{42} &= \beta + c_1 \\ c_{43} &= \alpha + b_1 + c_2 \\ c_{44} &= 1 + a_1 + b_2 + c3. \end{aligned}$$

Solving the equation $C_T \equiv 0$, we get

$$a_1 = -6; b_1 = -3, b_2 = 11; c_1 = -1; c_2 = 2; c_3 = -6.$$

Hence the logarithmic homogeneous differential equation of 4^{th} order is given by

$$A_E Y_x = 0,$$

i.e.,

$$e^{-4x} y^{iv} + (\alpha e^{-3x} - 6e^{-4x}) y''' + (\beta e^{-2x} - 3\alpha e^{-3x} + 11e^{-4x}) y'' \\ + (\gamma e^{-x} - \beta e^{-2x} + 2e^{-3x} - 6e^{-4x}) y' + \delta y = 0.$$

Note 4.1. Suppose a dynamic problem depending on actual time variable (T) changing logarithmically with respect to time t. Therefore, we have coined a term called *Logarithmic Dynamic/Differential Equation* varying with respect to $T = \ln t$, $t \neq 0$. The equation can be converted to an Euler-Cauchy equation changing with respect to time t.

Let $T = \ln t$, $t \neq 0$. Denoting $y', y'', \cdots$ as derivative of y with respect to T and $\dot{y}, \ddot{y}, \cdots$ as derivative of y with respect to t, we have

$$\begin{aligned} t\dot{y} &= y', \\ t^2\ddot{y} &= y'' - y', \\ t^3\dddot{y} &= y''' - 3y'' - 2y', \\ t^4\ddddot{y} &= y^{iv} - 6y''' + 11y'' + 18y', \\ &\vdots \end{aligned}$$

The homogeneous logarithmic differential equations can be written as Euler-Cauchy equation as follows.

$$\begin{aligned} y'' + (a-1)y' + by = 0 \quad &\Leftrightarrow \quad t^2\ddot{y} + at\dot{y} + by = 0; \\ y''' + (a-3)y'' + (b-a-2)y' + cy = 0 \quad &\Leftrightarrow \quad t^3\dddot{y} + at^2\ddot{y} + bt\dot{y} + cy = 0; \end{aligned}$$

$$\begin{aligned} & y^{iv} + (a-6)y''' + (11 - 3a + b)y'' \\ & \quad +(18 - 2a - b + c)y' + dy = 0 \\ \Leftrightarrow \quad & t^4\ddddot{y} + at^3\dddot{y} + bt^2\ddot{y} + dt\dot{y} + cy = 0 \end{aligned}$$

5 Sufficient Conditions for all Solutions to be Asymptotic to Polynomials at Infinity

For any $\Omega \subseteq \mathbb{R}$, let $y \in C^2(\Omega)$, the family of twice differentiable function on Ω. The equation

$$e^{-2x}y'' + \left(ae^{-x} - e^{-2x}\right)y' + by = 0$$

can be written as

$$-y'' = f(x, y, y')$$

where

$$f(x, y, y') = bye^{2x} + \left(ae^{x} - 1\right)y'$$

The sufficient conditions for all the solutions to be asymptotic to polynomials at infinity is studied in the following proposition.

Proposition 5.1. *Let*

$$|f(x, z_0, z_1)| \leq \sum_{k=0}^{1} p_k(x) g_k\left(\frac{|z_k|}{x_{1-k}}\right) + q(x)$$

for all $(x, z_0, z_1) \in [-\infty, x_0) \times \mathbb{R}^2$ *where* p_k, $k = 0, 1$ *and* q *are nonnegative continuous real-valued functions on* $(-\infty, x_0]$ *such that*

$$\int_{-\infty}^{x_0} p_k(x)dx < \infty (k = 0, 1,) \quad \textit{and} \quad \int_{-\infty}^{x_0} q(x)dx < \infty$$

and $g_k(k = 0, 1)$ *are continuous real-valued functions on* $(-\infty, 0]$*, which are positive and increasing on* $(-\infty, 0]$ *and such that*

$$\int_{-\infty}^{1} \frac{dz}{\sum_{k=0}^{1} g_k(z)} = \infty.$$

Then every solution x *on an interval* $(-\infty, T]$*,* $T \le x_0$*, of the differential equation* $(*)$ *satisfies*

$$y^{(j)}(x) = \frac{c}{(1-j)\,!} x_{1-j} + o(x_{1-j}) \; as \, x \to -\infty \, (j = 0, 1),$$

where c *is some real number (depending on the solution* y*).*

Proof. Since

$$f(x, z_0, z_1) = bz_0 \exp(x) + (ae^x - 1)\, z_1,$$

by Proposition 3.1 [9], every solution y on an interval $(-\infty, T]$, $T \le x_0$, of the differential equation $(*)$ with $n = 2$ satisfies

$$y^{(j)}(x) = \frac{c}{(1-j)\,!} x_{1-j} + o(x_{1-j}) \, \text{as} \, x \to -\infty \, (j = 0, 1),$$

where c is some real number (depending on the solution y). This completes the proof. □

Before closing this section, we note that it is especially an interesting problem to study the n-th order $(n > 1)$ nonlinear delay differential equation

$$y^{(n)}(x) = f(x, y(x - \tau_0(x)), y'(x - \tau_1(x)), ..., y^{(N)}(x - \tau_N(x))), \quad x \ge x_0 > 0,$$

obtained from (4.3) where $\tau_k(k = 0, 1, \cdots, N)$ are nonnegative continuous real-valued functions on $[x_0, \infty)$ such that $\lim_{x\to\infty} [x - \tau_k(x)] = \infty (k = 0, 1, ..., N)$.

6 Conclusion

We have construct a new class of second order differential equation at infinity which can be help to solve the problems arises in physical sciences and engineering. To solve the problem, we generate the auxiliary equation via a pre-auxiliary equation and obtain the general solution of it. We express the higher order of the differential equation in a matrix form which can be helpful to study the theory of variational inequalities and complementarities. We prove the sufficient condition of the solutions for the 2nd order differential equation under certain conditions for existence of the problem.

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Competing Interests

The author declares that no competing interests exist.

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