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Tridiagonal Matrices via k-Balancing Number

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Abstract

In this paper, we give some relations in terms of $k-$Balancing number which generalize some well known results concerning the relation between the determinant and Chebyshev polynomials which is due to tridiagonal matrix $B_{(n)}(k)$. Also for the other tridiagonal matrix $W_{(n)}(k)$, we deduce the cofactor matrix of it then we find another relations for $k-$Balancing number.

Keywords: Tridiagonal matrix; balancing number; cofactor matrix; determinant.

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1 Preliminaries

Recently, *balancing numbers* $n \in \mathbb{Z}^+$ as solutions of the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r) \tag{1.1}$$

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for some positive integer r which is called *balancer* or *cobalancing number.* For example $6, 35, 204$, 1189 and 6930 are balancing numbers with balancers $2, 14, 84, 492$ and 2870, respectively. If n is a balancing number with balancer r, then from (1.1) one has $\frac{(n-1)n}{2} = rn + \frac{r(r+1)}{2}$ and so

$$r = \frac{-(2n+1)+\sqrt{8n^2+1}}{2} \text{ and } n = \frac{2r+1+\sqrt{8r^2+8r+1}}{2}. \tag{1.2}$$

Let B_n denote the n^{th} balancing number and let b_n denote the n^{th} cobalancing number. Then $B_{n+1} = 6B_n - B_{n-1}$ and $b_{n+1} = 6b_n - b_{n-1} + 2$ for $n \geq 2$, where $B_1 = 1, B_2 = 6, b_1 = 0$ and $b_2 = 2$. From (1.2), we see that B_n is a balancing number iff $8B_n^2 + 1$ is a perfect square and b_n is a cobalancing number iff $8b_n^2 + 8b_n + 1$ is a perfect square. So we set

$$C_n = \sqrt{8B_n^2+1} \text{ and } c_n = \sqrt{8b_n^2+8b_n+1}$$

which are called the n^{th} *Lucas-balancing number* and n^{th} *Lucas-cobalancing number*, respectively [1], [2] and [3].

In [4] they generalized the theory of balancing numbers to numbers defined as: Let $y, k, l \in \mathbb{Z}^+$ such that $y \geq 4$. Then a positive integer x such that $x \leq y - 2$ is called a $(k, l)-$power numerical center for y if $1^k + \cdots + (x-1)^k = (x+1)^l + \cdots + (y-1)^l$. They derived some algebraic relation on it.

Because of the concept of the balancing numbers; we generalized the balancing numbers to $k-$balancing numbers: B_n^k denote the n $^{\text{th}}$ $k-$balancing number, b_n^k denote the n^{th} $k-$ cobalancing number, C_n^k denote the n^{th} $k-$Lucas balancing number and c_n^k denote the n^{th} $k-$Lucas cobalancing number which are the numbers defined by

$$\begin{aligned}
&B_0^k = 0, B_1^k = 1,\ B_{n+1}^k = 6kB_n^k - B_{n-1}^k \text{ for } n \geq 1 \\
&b_1^k = 0, b_2^k = 2, b_{n+1}^k = 6kb_n^k - b_{n-1}^k + 2 \text{ for } n \geq 2 \\
&C_0^k = 1, C_1^k = 3, C_{n+1}^k = 6kC_n^k - C_{n-1}^k \text{ for } n \geq 1 \\
&c_1^k = 1, c_2^k = 7, c_{n+1}^k = 6kc_n^k - c_{n-1}^k \text{ for } n \geq 2
\end{aligned} \tag{1.3}$$

for some positive integer $k \geq 1$, respectively. Also Binet formulas for $k-$balancing numbers are

$$\begin{aligned}
B_n^k &= \frac{\alpha^n - \beta^n}{2\sqrt{9k^2-1}} \\
b_n^k &= \frac{(\alpha+1)\alpha^{n-1} + (\beta+1)\beta^{n-1} - 6k - 2}{2(9k^2-1)} \\
C_n^k &= \frac{(3-\beta)\alpha^n - (3-\alpha)\beta^n}{2\sqrt{9k^2-1}} \\
c_n^k &= \frac{(7\alpha-1)\alpha^{n-2} - (7\beta-1)\beta^{n-2}}{2\sqrt{9k^2-1}}
\end{aligned}$$

for $n \geq 1$, where $\alpha = 3k + \sqrt{9k^2-1}$ and $\beta = 3k - \sqrt{9k^2-1}$.

A tridiagonal matrix is a matrix that is both upper and lower Hessenberg matrix. A general tridiagonal matrix is not necessarily symmetric or Hermitian, many of those that arise when solving linear algebra problems have one of these properties. Furthermore, if a real tridiagonal matrix A satisfies $a_{k,k+1}a_{k+1,k} > 0$ for all k, so that the signs of its entries are symmetric, then it is similar to a Hermitian matrix, by a diagonal change of basis matrix. Hence, its eigenvalues are real. If we replace the the strict inequality by $a_{k,k+1}a_{k+1,k} \geq 0$, then the eigenvalue are guaranteed to be real, but the matrix need no longer be similar to a Hermitian matrix.

$T_{(n)}$ be a family of $n \times n$ tridiagonal matrices, where

$$T_{(n)} = \begin{bmatrix} a_{1,1} & a_{1,2} & & & & \\ a_{2,1} & a_{2,2} & a_{2,3} & & & \\ & a_{3,2} & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & a_{n-1,n} \\ & & & & a_{n,n-1} & a_{n,n} \end{bmatrix}$$

(for further details see [5], [6] and [7]).

Theorem 1.1. *The determinants of $T_{(n)}$ are defined to be with initial values*

$$\begin{aligned} \det(T_{(1)}) &= a_{1,1} \\ \det(T_{(2)}) &= a_{2,2}a_{1,1} - a_{2,1}a_{1,2} \\ \det(T_{(n)}) &= a_{n,n}\det(T_{(n-1)}) - a_{n,n-1}a_{n-1,n}\det(T_{(n-2)}) \end{aligned}$$

for $n \geq 3$, where $a_{i,j}$ its i^{th} row and j^{th} column are non-zero integers such that $D = a_{n,n}^2 - 4a_{n,n-1}a_{n-1,n} \neq 0$ (For the proof see [8]).

The characteristic equation of the recurrence relation is $x^2 - a_{n,n}x + a_{n,n-1}a_{n-1,n} = 0$ and hence the roots of it are $\alpha = \frac{a_{n,n}+\sqrt{D}}{2}$ and $\beta = \frac{a_{n,n}-\sqrt{D}}{2}$. So their Binet's formulas are $\det(T_{(n)}) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$.

Let A be an $n \times n$ matrix and let M_{ij} be the $(n-1)\times(n-1)$ matrix obtained with deleting the i^{th} row and j^{th} column of the matrix then computing the determinant of the remaining matrix after deleting the row and column. Also finding the cofactors of a matrix, just use the minor $C_{ij} = (-1)^{i+j}M_{ij}$ where M_{ij} is the minor in the i^{th} row and j^{th} position of the matrix. The adjoint of a matrix denoted by $adj(A)$, to find the adjoint of a matrix transpose the cofactor matrix.

For finding inverses using the adjoint

$$A^{-1} = \frac{1}{\det(A(n))} adj(A).$$

The cofactors feature prominently in Laplace's formula for the expansion of determinants, which is a method of computing larger determinants in terms of smaller ones. Given the $n \times n$ matrix (a_{ij}), the determinant of A can be written as the sum of the cofactors of any row or column of the matrix multiplied by the entries that generated them. The cofactor expansion along the j^{th} column gives:

$$\det(A(n)) = a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j} + \cdots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}.$$

The cofactor expansion along the i^{th} row gives:

$$\det(A(n)) = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} + \cdots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}.$$

The inverse of an invertible matrix by computing its cofactors by using Cramer's rule. The matrix formed by all of the cofactors of a square matrix A is called the cofactor matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}.$$

Then the inverse of A is the transpose of the cofactor matrix times the inverse of the determinant of A:

$$A^{-1} = \frac{1}{\det(A(n))} C^T \tag{1.4}$$

which are discussed by [9] and [10].

The Chebyshev polynomials are a sequence of orthogonal polynomials appearing in approximation theory and they have countless applications. The first and second kind of Chebyshev polynomials satisfy the same recurrence relations. The Chebyshev polynomials of the first and second kind are defined by $T_0(x) = 1$, $T_1(x) = x$ for $n \geq 2$ and $U_0(x) = 1$, $U_1(x) = 2x$ for $n \geq 2$,

$$\begin{aligned} T_n(x) &= 2xT_{n-1}(x) - T_{n-2}(x) \\ U_n(x) &= 2xU_{n-1}(x) - U_{n-2}(x) \end{aligned} \tag{1.5}$$

respectively. Also Chebyshev polynomials satisfying $T_n(\cos\theta) = \cos n\theta$ for $n = 0, 1, 2, \cdots$ and $U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$ for $n = 0, 1, 2, \cdots$. They have the determinant representation

$$T_n(x) = \begin{vmatrix} x & z & & & \\ y & 2x & z & & \\ & y & 2x & \ddots & \\ & & \ddots & \ddots & z \\ & & & y & 2x \end{vmatrix} \text{ and } U_n(x) = \begin{vmatrix} 2x & z & & & \\ y & 2x & z & & \\ & y & 2x & \ddots & \\ & & \ddots & \ddots & z \\ & & & y & 2x \end{vmatrix}$$

where $yz = 1$. As discussed elsewhere [11].

Furthermore there are more relations between $T_n(x)$ and $U_n(x)$, for example

$$\begin{aligned} 2T_n(x) &= U_n(x) - U_{n-2}(x) \\ T_n(x) &= U_n(x) - xU_{n-1}(x) \\ T_{n+1}(x) &= xT_n(x) - (1 - x^2)U_{n-1}(x) \\ U_n(x) &= 2\sum_{j\ odd}^{n} T_j(x), \text{ where } n \text{ is odd} \\ U_n(x) &= 2\sum_{j\ even}^{n} T_j(x) - 1, \text{ where } n \text{ is even.} \end{aligned} \tag{1.6}$$

2 Main Results

$B_{(n)}(k)$ and $W_{(n)}(k)$ be a family of $n \times n$ tridiagonal matrices for $k-$balancing number are defined to be

$$W_{(n)}(k) = \begin{bmatrix} 36k & 6 & & & & \\ 1 & 6k & 1 & & & \\ & 1 & 6k & 1 & & \\ & & 1 & \ddots & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 1 & 6k \end{bmatrix}_{n\times n} \tag{2.1}$$

and

$$B_{(n)}(k) = \begin{bmatrix} 6k & 1-18k^2 & & & & \\ 1 & 3k & 1 & & & \\ & 1 & 6k & 1 & & \\ & & 1 & \ddots & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 1 & 6k \end{bmatrix}_{n\times n}. \tag{2.2}$$

So we obtain six times of the $(n+1)-th$, $k-$balancing number from the determinants of the matrices

$$\begin{aligned} \det(W_{(1)}(k)) &= 36k = 6B_2^k \\ \det(W_{(2)}(k)) &= 216k^2 - 6 = 6B_3^k \\ \det(W_{(3)}(k)) &= 6k(216k^2-6) - 36k = 6B_4^k \\ &\vdots \\ \det(W_{(n)}(k)) &= 6k\det(W_{(n-1)}(k)) - \det(W_{(n-2)}(k)) = 6B_{n+1}^k \end{aligned}$$

and

$$\begin{aligned} \det(B_{(1)}(k)) &= B_2^k \\ \det(B_{(2)}(k)) &= B_3^k \\ \det(B_{(3)}(k)) &= B_4^k \\ &\vdots \\ \det(B_{(n)}(k)) &= B_{n+1}^k. \end{aligned} \tag{2.3}$$

Here one may wonder why we choice these two tridiagonal matrices, because note that $\det(B_{(n)}(k)) = B_{n+1}^k$ and $\det(W_{(n)}(k)) = 6B_{n+1}^k$ in other words both of the determinant $B_{(n)}(k)$ and the determinant of the $W_{(n)}(k)$ may be expressed in $k-$balancing number, even 6 times each. So we have to need two tridiagonal matrices which is denoted with $k-$balancing number.

Also we define the odd $k-$balancing number from the determinant of the matrices

$$O_{(n)}(k) = \begin{bmatrix} 36k^2-1 & 1 & & & & & \\ 1 & 36k^2-2 & 1 & & & & \\ & 1 & 36k^2-2 & 1 & & & \\ & & 1 & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & 1 \\ & & & & & 1 & 36k^2-2 \end{bmatrix}.$$

because $\det(O_{(n)}(k)) = B^k_{2n+1}$, which is $\{1, 36k^2-1, 1296k^4-108k^2+1, \cdots\}$ the odd $k-$balancing number. Finally we define the even $k-$balancing number

$$E_{(n)}(k) = \begin{bmatrix} 6k & 0 & & & & & \\ 1 & 36k^2-2 & 1 & & & & \\ & 1 & 36k^2-2 & 1 & & & \\ & & 1 & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & 1 \\ & & & & & 1 & 36k^2-2 \end{bmatrix}$$

which is $\det(E_{(n)}(k)) = B^k_{2n}$.

Lemma 2.1. *If $B_{(n)}(k)$ are tridiagonal matrices of the form (2.2), then the determinant of $B_{(n)}(k)$ is*

$$\det(B_{(n)}(k)) = 6kT_{n-1}(3k) - (1-18k^2)U_{n-2}(3k)$$

with Chebyshev polynomials and the characteristic polynomial of A is

$$p_{B_{(n)}(k)}(\lambda) = (\lambda-6k)T_{n-1}(\lambda-3k) - (\lambda-1+18k^2)U_{n-2}(\lambda-3k) \tag{2.4}$$

Proof. Let $A_{(n)}(k)$ and $C_{(n)}(k)$ be a tridiagonal matrix, so

$$\det(A_{(n)}(k)) = \begin{vmatrix} 3k & 1 & & & \\ 1 & 6k & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 6k \end{vmatrix}_{n\times n} \quad and$$

$$\det(C_{(n)}(k)) = \begin{vmatrix} 6k & 1 & & & \\ 1 & 6k & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 6k \end{vmatrix}_{n\times n}.$$

In the light of the Laplace expression we expand the $B_{(n)}(k)$ with terms of the first kind of Chebyshev polynomial $T_n(x)$ and second kind of Chebyshev polynomial $U_n(x)$. By expanding the determinant with the first columb, we have

$$\det(B_{(n)}(k)) \quad = \quad \begin{vmatrix} 6k & 1-18k^2 & & & & \\ 1 & 3k & 1 & & & \\ & 1 & 6k & 1 & & \\ & & 1 & \ddots & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 1 & 6k \end{vmatrix}_{n\times n}$$

$$
=6\text{k}\begin{vmatrix} 3k & 1 & & & \\ 1 & 6k & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 6k \end{vmatrix}_{(n-1)\times(n-1)} - \begin{vmatrix} 1-18k^2 & 0 & & & \\ 1 & 6k & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 6k \end{vmatrix}_{(n-1)\times(n-1)}
$$

$$
=6\text{k}\begin{vmatrix} 3k & 1 & & & \\ 1 & 6k & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 6k \end{vmatrix}_{(n-1)\times(n-1)} - (1-18k^2)\begin{vmatrix} 6k & 1 & & & \\ 1 & 6k & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 6k \end{vmatrix}_{(n-2)\times(n-2)}
$$

$$
=6\text{k}\det(A_{(n-1)}(k)) - (1-18k^2)\det(C_{(n-2)}(k)).
$$

Determinants in the expression are obtained a special case of the Chebyshev polynomials first and second kind in (1.5). So we get

$$
\det(B_{(n)}(k)) = 6kT_{n-1}(3k) - (1-18k^2)U_{n-2}(3k).
$$

The characteristic polynomial are

$$
p_{B_{(n)}(k)}(\lambda) = \det(\lambda I - B_{(n)}(k)) = (\lambda - 6k)T_{n-1}(\lambda - 3k) - (\lambda - 1 + 18k^2)U_{n-2}(\lambda - 3k)
$$

where I is the identity matrix. □

Consequently we can give the following theorem, $B_{(n)}(k)$ satisfies.

Theorem 2.2. *The eigenvalues of the tridiagonal matrix $B_{(n)}(k)$ are*

$$
\lambda_i = 3k + \cos\frac{i\pi}{n+1}, (i = 1, 2, \cdots, n)
$$

and the n^{th} $k-$balancing number is denoted by

$$
B_n^k = \prod_{i=1}^{n-1}\left(3k + \cos\frac{i\pi}{n}\right).
$$

Proof. From recurrence relations of $U_n(x)$ and (1.6)

$$
\begin{aligned}
\det(B_{(n)}(k)) &= 6kT_{n-1}(3k) - (1-18k^2)U_{n-2}(3k) \\
&= 3kU_{n-1}(3k) - 3kU_{n-3}(3k) - (1-18k^2)U_{n-2}(3k) \\
&= U_n(3k) + 3k(6kU_{n-2}(3k) - U_{n-3}(3k)) - 3kU_{n-1}(3k) \\
&= U_n(3k).
\end{aligned}
$$

Hence the eigenvalues of $B_{(n)}(k)$ can be obtained through computing the zeros of the characteristic polynomial (2.4). In view of the roots of $U_n(x) = 0$ are $\theta_i = \frac{i\pi}{n+1}$, $(i = 1, 2, \cdots, n)$ or equally $x_i = \cos\theta_i = \cos\frac{i\pi}{n+1}$ and the eigenvalues of $B_{(n)}(k)$ are

$$
\begin{aligned}
p_{B_{(n)}(k)}(\lambda_i) &= \det(\lambda_i I - B_{(n)}(k)) \\
&= U_n(\lambda_i I - 3k)
\end{aligned}
$$

So

$$
\lambda_i = 3k + \cos\frac{i\pi}{n+1}, (i = 1, 2, \cdots, n)
$$

Moreover we know the equation $\det(B_{(n)}(k)) = B^k_{n+1}$ from (2.3) and the definition of the determinant of the matrix

$$B^k_n = \prod_{i=1}^{n-1}\left(3k + \cos\frac{i\pi}{n}\right)$$

result is clear. $\square$

From this result, we can obtain the following conclusions.

Corollary 2.3. *Let B^k_n denote the n^{th} $k-$Balancing number and $B_{(n)}(k)$ is a tridiagonal matrices, then*

1. $\det(B_{(n)}(k)) = \prod_{i=1}^{n-1}\left(3k + \cos\frac{i\pi}{n}\right)$
2. *If* $3k \neq -\cos\frac{i\pi}{n+1}, (i = 1, 2, \cdots, n-1)$ *then* $B_{(n)}(k)$ *is invertible*
3. $\det(W_{(n)}(k)) = 6\prod_{i=1}^{n-1}\left(3k + \cos\frac{i\pi}{n}\right)$.

Now we can obtain the relation between the matrix $W_{(n)}(k)$ and $B_{(n)}(k)$ given by $k-$Balancing number.

In order to determine the inverse of matrix $W_{(n)}(k)$, we have to formulate the cofactor matrix of $W_{(n)}(k)$. For this reason, we can give $C_{(n)}(k)$ cofactor matrix of $W_{(n)}(k)$ are expressed in terms of $k-$Balancing number, which can be proved by induction on n.

Lemma 2.4. *Let B^k_n denote the n^{th} k-balancing number and $W_{(n)}(k)$ are the tridiagonal matrices. $C_{(n-1)}(k)$, the cofactor matrices of $W_{(n)}(k)$ whose elements are given by*

$$m_{nj} = \begin{cases} 6B^k_j & \text{if } 2 \leq j \leq n \text{ is even} \\ -6B^k_j & \text{if } 3 \leq j \leq n \text{ is odd} \end{cases}, m_{in} = \begin{cases} 6B^k_i & \text{if } 2 \leq i \leq n-1 \text{ is even} \\ -6B^k_i & \text{if } 3 \leq i \leq n-1 \text{ is odd} \end{cases}$$
$$m_{i1} = \begin{cases} -B^k_{n-(i-1)} & \text{if } 2 \leq i \leq n-1 \text{ is even} \\ B^k_{n-(i-1)} & \text{if } 1 \leq i \leq n-1 \text{ is odd} \end{cases}, m_{1j} = \begin{cases} -6B^k_{n-(j-1)} & \text{if } 2 \leq j \leq n-1 \text{ is even} \\ 6B^k_{n-(j-1)} & \text{if } 3 \leq j \leq n-1 \text{ is odd} \end{cases}$$

and the other terms

$$m_{ij} = \begin{cases} -6B^k_iB^k_{n-(j-1)} & \text{if } j > i, (-1)^{i+j} = -1 \\ 6B^k_iB^k_{n-(j-1)} & \text{if } j > i, (-1)^{i+j} = 1 \end{cases}, m_{ij} = \begin{cases} -6B^k_jB^k_{n-(i-1)} & \text{if } j < i, (-1)^{i+j} = -1 \\ 6B^k_jB^k_{n-(i-1)} & \text{if } j < i, (-1)^{i+j} = 1 \end{cases}$$
$$m_{ij} = 6B^k_iB^k_{n-(j-1)} \text{ if } i = j(ij \neq 11 \text{ and } ij \neq nn), m_{1n} = -6B^k_1$$

with j is the j^{th} columb and i is the i^{th} row of $W_{(n)}(k)$ for $n \geq 4$ is even and

$$m_{nj} = \begin{cases} -6B^k_j & \text{if } 2 \leq j \leq n \text{ is even} \\ 6B^k_j & \text{if } 3 \leq j \leq n \text{ is odd} \end{cases}, m_{in} = \begin{cases} -6B^k_i & \text{if } 2 \leq i \leq n-1 \text{ is even} \\ 6B^k_i & \text{if } 3 \leq i \leq n-1 \text{ is odd} \end{cases}$$
$$m_{i1} = \begin{cases} -B^k_{n-(i-1)} & \text{if } 2 \leq i \leq n-1 \text{ is even} \\ B^k_{n-(i-1)} & \text{if } 1 \leq i \leq n-1 \text{ is odd} \end{cases}, m_{1j} = \begin{cases} -6B^k_{n-(j-1)} & \text{if } 2 \leq j \leq n \text{ is even} \\ 6B^k_{n-(j-1)} & \text{if } 3 \leq j \leq n \text{ is odd} \end{cases}$$

and the other terms

$$m_{ij} = \begin{cases} -6B^k_iB^k_{n-(j-1)} & \text{if } j > i, (-1)^{i+j} = -1 \\ 6B^k_iB^k_{n-(j-1)} & \text{if } j > i, (-1)^{i+j} = 1 \end{cases}, m_{ij} = \begin{cases} -6B^k_jB^k_{n-(i-1)} & \text{if } j < i, (-1)^{i+j} = -1 \\ 6B^k_jB^k_{n-(i-1)} & \text{if } j < i, (-1)^{i+j} = 1 \end{cases}$$
$$m_{ij} = 6B^k_iB^k_{n-(j-1)} \text{ if } i = j(ij \neq 11 \text{ and } ij \neq nn), \; m_{1n} = 6B^k_1$$

with j is the j^{th} columb and i is the i^{th} row of $W_{(n)}(k)$ for $n \geq 5$ is odd.

Corollary 2.5. *Let B_n^k denote the n^{th} k-balancing number and $C_{(n)}(k)$ are the cofactor matrices of $W_{(n)}(k)$,*

1. *The determinant of $C_{(n)}(k)$ is the $(n-1)^{th}$ power of the six times of the terms of the $(n+1)^{th}$ $k-$balancing number*
$$\det(C_{(n)}(k)) = (6B_{n+1}^k)^{n-1}$$
2. *The n^{th} power of the six times of the terms of the $(n+3)^{th}$ $k-$balancing number is denoted by the ratio of the determinants of the $C_{(n+2)}(k)$ and $W_{(n+2)}(k)$*
$$\frac{\det(C_{(n+2)}(k))}{\det(W_{(n+2)}(k))} = (6B_{n+3}^k)^n.$$

Theorem 2.6. *Let B_n^k denote the n^{th} k-balancing number and $C_{(n)}(k)$ are the cofactor matrices of $W_{(n)}(k)$, then the inverse of matrix $W_{(n)}(k)$ is denoted by*

$$m_{nj} = \begin{cases} \frac{1}{B_{n+1}^k} B_j^k & \text{if } 2 \leq j \leq n \text{ is even} \\ -\frac{1}{B_{n+1}^k} B_j^k & \text{if } 3 \leq j \leq n \text{ is odd} \end{cases}, m_{in} = \begin{cases} \frac{1}{B_{n+1}^k} B_i^k & \text{if } 2 \leq i \leq n-1 \text{ is even} \\ -\frac{1}{B_{n+1}^k} B_i^k & \text{if } 3 \leq i \leq n-1 \text{ is odd} \end{cases}$$
$$m_{i1} = \begin{cases} -\frac{1}{6B_{n+1}^k} B_{n-(i-1)}^k & \text{if } 2 \leq i \leq n-1 \text{ is even} \\ \frac{1}{6B_{n+1}^k} B_{n-(i-1)}^k & \text{if } 1 \leq i \leq n-1 \text{ is odd} \end{cases}, m_{1j} = \begin{cases} -\frac{1}{B_{n+1}^k} B_{n-(j-1)}^k & \text{if } 2 \leq j \leq n-1 \text{ is even} \\ \frac{1}{B_{n+1}^k} B_{n-(j-1)}^k & \text{if } 3 \leq j \leq n-1 \text{ is odd} \end{cases}$$

and the other terms

$$m_{ij} = \begin{cases} -\frac{1}{B_{n+1}^k} B_i^k B_{n-(j-1)}^k & \text{if } j > i, (-1)^{i+j} = -1 \\ \frac{1}{B_{n+1}^k} B_i^k B_{n-(j-1)}^k & \text{if } j > i, (-1)^{i+j} = 1 \end{cases}, m_{ij} = \begin{cases} -\frac{1}{B_{n+1}^k} B_j^k B_{n-(i-1)}^k & \text{if } j < i, (-1)^{i+j} = -1 \\ \frac{1}{B_{n+1}^k} B_j^k B_{n-(i-1)}^k & \text{if } j < i, (-1)^{i+j} = 1 \end{cases}$$
$$m_{ij} = \frac{1}{B_{n+1}^k} B_i^k B_{n-(j-1)}^k \text{ if } i = j (ij \neq 11 \text{ and } ij \neq nn), m_{1n} = -\frac{1}{B_{n+1}^k} B_1^k$$

with j is the j^{th} columb and i is the i^{th} row of $W_{(n)}(k)$ for $n \geq 4$ is even and

$$m_{nj} = \begin{cases} 6B_j^k & \text{if } 2 \leq j \leq n \text{ is even} \\ -6B_j^k & \text{if } 3 \leq j \leq n \text{ is odd} \end{cases}, m_{in} = \begin{cases} 6B_i^k & \text{if } 2 \leq i \leq n-1 \text{ is even} \\ -6B_i^k & \text{if } 3 \leq i \leq n-1 \text{ is odd} \end{cases}$$
$$m_{i1} = \begin{cases} -B_{n-(i-1)}^k & \text{if } 2 \leq i \leq n-1 \text{ is even} \\ B_{n-(i-1)}^k & \text{if } 1 \leq i \leq n-1 \text{ is odd} \end{cases}, m_{1j} = \begin{cases} -6B_{n-(j-1)}^k & \text{if } 2 \leq j \leq n-1 \text{ is even} \\ 6B_{n-(j-1)}^k & \text{if } 3 \leq j \leq n-1 \text{ is odd} \end{cases}$$

and the other terms

$$m_{ij} = \begin{cases} -6B_i^k B_{n-(j-1)}^k & \text{if } j > i, (-1)^{i+j} = -1 \\ 6B_i^k B_{n-(j-1)}^k & \text{if } j > i, (-1)^{i+j} = 1 \end{cases}, m_{ij} = \begin{cases} -6B_j^k B_{n-(i-1)}^k & \text{if } j < i, (-1)^{i+j} = -1 \\ 6B_j^k B_{n-(i-1)}^k & \text{if } j < i, (-1)^{i+j} = 1 \end{cases}$$
$$m_{ij} = 6B_i^k B_{n-(j-1)}^k \text{ if } i = j (ij \neq 11 \text{ and } ij \neq nn), m_{1n} = -6B_1^k$$

with j is the j^{th} columb and i is the i^{th} row of $W_{(n)}(k)$ for $n \geq 5$ is odd.

Proof. We know the truth from (1.4) that

$$\begin{aligned}(W_{(n)}(k))^{-1} &= \frac{1}{\det(W_{(n)}(k))}C_{(n)}(k)^T\\ (W_{(n)}(k))^{-1} &= \frac{1}{6B^k_{n+1}}C_{(n)}(k)^T\end{aligned}$$

We know its transpoze of $C_{(n)}(k)$ from Lemma 2.4

$$\begin{aligned}m_{nj} &= \begin{cases} 6B^k_j & \text{if } 2\leq j\leq n \text{ is even}\\ -6B^k_j & \text{if } 3\leq j\leq n \text{ is odd}\end{cases}, m_{in} = \begin{cases} 6B^k_i & \text{if } 2\leq i\leq n-1 \text{ is even}\\ -6B^k_i & \text{if } 3\leq i\leq n-1 \text{ is odd}\end{cases}\\ m_{i1} &= \begin{cases} -B^k_{n-(i-1)} & \text{if } 2\leq i\leq n-1 \text{ is even}\\ B^k_{n-(i-1)} & \text{if } 1\leq i\leq n-1 \text{ is odd}\end{cases}, m_{1j} = \begin{cases} -6B^k_{n-(j-1)} & \text{if } 2\leq j\leq n-1 \text{ is even}\\ 6B^k_{n-(j-1)} & \text{if } 3\leq j\leq n-1 \text{ is odd}\end{cases}\end{aligned}$$

and the other terms

$$\begin{aligned}m_{ij} &= \begin{cases} -6B^k_iB^k_{n-(j-1)} & \text{if } j>i, (-1)^{i+j}=-1\\ 6B^k_iB^k_{n-(j-1)} & \text{if } j>i, (-1)^{i+j}=1\end{cases}, m_{ij} = \begin{cases} -6B^k_jB^k_{n-(i-1)} & \text{if } j<i, (-1)^{i+j}=-1\\ 6B^k_jB^k_{n-(i-1)} & \text{if } j<i, (-1)^{i+j}=1\end{cases}\\ m_{ij} &= 6B^k_iB^k_{n-(j-1)} \text{ if } \; i=j(ij\neq 11 \text{ and } ij\neq nn), m_{1n}=-6B^k_1\end{aligned}$$

with j is the j^{th} columb and i is the i^{th} row of $W_{(n)}(k)$ for $n \geq 4$ is even and

$$\begin{aligned}m_{nj} &= \begin{cases} -6B^k_j & \text{if } 2\leq j\leq n \text{ is even}\\ 6B^k_j & \text{if } 3\leq j\leq n \text{ is odd}\end{cases}, m_{in} = \begin{cases} -6B^k_i & \text{if } 2\leq i\leq n-1 \text{ is even}\\ 6B^k_i & \text{if } 3\leq i\leq n-1 \text{ is odd}\end{cases}\\ m_{i1} &= \begin{cases} -B^k_{n-(i-1)} & \text{if } 2\leq i\leq n-1 \text{ is even}\\ B^k_{n-(i-1)} & \text{if } 1\leq i\leq n-1 \text{ is odd}\end{cases}, m_{1j} = \begin{cases} -6B^k_{n-(j-1)} & \text{if } 2\leq j\leq n \text{ is even}\\ 6B^k_{n-(j-1)} & \text{if } 3\leq j\leq n \text{ is odd}\end{cases}\end{aligned}$$

and the other terms

$$\begin{aligned}m_{ij} &= \begin{cases} -6B^k_iB^k_{n-(j-1)} & \text{if } j>i, (-1)^{i+j}=-1\\ 6B^k_iB^k_{n-(j-1)} & \text{if } j>i, (-1)^{i+j}=1\end{cases}, m_{ij} = \begin{cases} -6B^k_jB^k_{n-(i-1)} & \text{if } j<i, (-1)^{i+j}=-1\\ 6B^k_jB^k_{n-(i-1)} & \text{if } j<i, (-1)^{i+j}=1\end{cases}\\ m_{ij} &= 6B^k_iB^k_{n-(j-1)} \text{ if } \; i=j(ij\neq 11 \text{ and } ij\neq nn),\; m_{1n}=6B^k_1\end{aligned}$$

with j is the j^{th} columb and i is the i^{th} row of $W_{(n)}(k)$ for $n \geq 5$ is odd.

So we product the terms of these matrice with $\frac{1}{6B^k_{n+1}}$, then we get the result. □

3 Conclusions

We proved some formulas differently to the traditional form by the $k-$balancing numbers concept. We discussed the relations between the $k-$balancing number and other results in this paper, for example trace, determinants, eigenvalues and so on. The two concepts balancing numbers and tridiagonal matrix in this work, have applications as in [12] and [13]. Therefore we obtained the applications to $k-$balancing numbers.

Some further investigations are as follows.

1. We consider only B_n^k defined from the determinant of the matrices, if possible we can discuss the other $k-$balancing numbers which are defined in (1.3) : b_n^k , C_n^k and c_n^k.
2. The results found in this paper can be used on the applications of $k-$balancing number. Also $k-$balancing numbers have connection between Pell numbers with the help of the Binet formulas.

Competing Interests

The author declares that no competing interests exist.

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