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The Equivalence Theorem for a K-Functional and a Modulus of Smoothness Constructed by a Singular Differential-Difference Operator on $\mathbb{R}$

R. F. Al Subaie[1] **and M. A. Mourou**[2*]

[1] *Department of Mathematics, College of Sciences for Girls, University of Dammam, P.O.Box 1982, Dammam 31441, Saudi Arabia.*

[2] *Department of Mathematics, Faculty of Sciences of Monastir, University of Monastir,* 5019, *Monastir, Tunisia.*

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Abstract

The purpose of this article is to establish the equivalence between a K-functional and a modulus of smoothness generated by a Dunkl type operator on the real line.

Keywords: Differential-difference operator; generalized fourier transform; generalized translation operator; K-functional; Modulus of smoothness.

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1 Introduction

Given a positive real number r and a positive integer m, the classical modulus of smoothness is defined for a function $f \in L^2(\mathbb{R})$ by

$$\omega_m(f, r) = \sup_{0<h\leq r} \|\Delta_h^m f\|_2,$$

Corresponding author: E-mail: mohamed_ali.mourou@yahoo.fr

where

$$\Delta_h^m f = \left(\tau^h - I\right)^m f, \tag{1.1}$$

I being the unit operator and τ^h stands for the usual translation operator given by $\tau^h f(x) = f(x+h)$. While the classical K-functional, introduced in [1], is defined by

$$K_m(f,r) = \inf\left\{\|f-g\|_2 + r\left\|D^m g\right\|_2 \,:\, g \in \mathcal{W}_2^m\right\},$$

where $D = d/dx$ and

$$\mathcal{W}_2^m = \left\{f \in L^2(\mathbb{R}) : D^j f \in L^2(\mathbb{R}),\ j = 1,2,...,m\right\}.$$

An outstanding result of the theory of approximation of functions on $\mathbb{R}$, which establishes the equivalence between modulus of smoothness and K-functionals, can be formulated as follows:

Theorem 1.1. *(see [2]) There are two positive constants c_1 and c_2 such that for all $f \in L^2(\mathbb{R})$ and $r > 0$:*

$$c_1\,\omega_m(f,r) \le\ K_m(f,r^m) \le c_2\,\omega_m(f,r).$$

Considerable attention has been devoted to discovering generalizations to new contexts for Theorem 1.1, see for instance [3]-[7]. The intention of this paper is to prove an analogue of Theorem 1.1 when in (1.1) the usual translation operators τ^h are substituted by certain generalized translation operators on $\mathbb{R}$ tied to the first-order singular differential-difference operator

$$\Lambda f(x) = f'(x) + \left(\gamma + \frac{1}{2}\right)\frac{f(x)-f(-x)}{x} + q(x)f(x),$$

where $\gamma > -1/2$ and q is a C^∞ real-valued odd function on $\mathbb{R}$. For $q = 0$, we retrieve the differential-difference operator

$$D_\gamma f(x) = f'(x) + \left(\gamma + \frac{1}{2}\right)\frac{f(x)-f(-x)}{x}, \tag{1.2}$$

which is referred to as the Dunkl operator with parameter $\gamma + 1/2$ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. Such operators have been introduced by Dunkl [8]-[10] in connection with a generalization of the classical theory of spherical harmonics. Besides its mathematical interest, the Dunkl operator D_γ has quantum-mechanical applications; it is naturally involved in the study of one-dimensional harmonic oscillators governed by Wigner's commutation rules [11]-[13].

In [14]-[15] the second author has initiated a completely new harmonic analysis related to the differential-difference operator Λ in which several analytic structures on $\mathbb{R}$ were generalized. The tools actually required for the discussion in the present paper, are essentially the Fourier transform and the translation operators linked to Λ. It is noted that the results stated in [4] may be recovered from those obtained in the present paper by simply taking $q = 0$.

2 Preliminaries

Throughout this section, we recapitulate some facts about harmonic analysis related to the differential-difference operator Λ. We cite here, as briefly as possible, only those properties really required for the discussion. For further details, we refer to [14]-[15]. From now on assume $\gamma > -1/2$.

The one-dimensional Dunkl kernel is defined by

$$e_\gamma(z) = j_\gamma(iz) + \frac{z}{2(\gamma+1)}j_{\gamma+1}(iz) \quad (z \in \mathbb{C}),$$

j_γ being the normalized spherical Bessel function of index γ given by

$$j_\gamma(z) = \Gamma(\gamma+1)\sum_{n=0}^{\infty}\frac{(-1)^n\ (z/2)^{2n}}{n!\ \Gamma(n+\gamma+1)} \quad (z\in\mathbb{C}).$$

The following properties collected from [3]-[4], [14] will play a key role in the sequel.

Lemma 2.1. *(i) For each $\lambda\in\mathbb{C}$, the function $e_\gamma(\lambda\cdot)$ is the unique solution of the differential-difference equation*

$$D_\gamma u = \lambda u, \quad u(0)=1. \tag{2.1}$$

(ii) For all $z\in\mathbb{C}$ and $n=0,1,...,$

$$\left|\frac{d^n}{dz^n}e_\gamma(z)\right| \le e^{|Rez|}. \tag{2.2}$$

(iii) There is $c_\gamma>0$ such that $|1-e_\gamma(ix)|\ge c_\gamma$ for all $x\in\mathbb{R}$ with $|x|\ge 1$.

(iv) For all $x\in\mathbb{R}-\{0\}$, $e_\gamma(ix)\neq 1$.

(v) For all $x\in\mathbb{R}$,

$$|1-e_\gamma(ix)|\le |x|.$$

Notation 2.1. *Put*

$$Q(x) = \exp\Big(-\int_0^x q(t)dt\Big), \quad x\in\mathbb{R}.$$

We denote by

- *$\mathcal{S}(\mathbb{R})$ the space of C^∞ functions f on $\mathbb{R}$, which are rapidly decreasing together with their derivatives, i.e., such that for all $m,n=0,1,...,$*

$$p_{m,n}(f) = \sup_{x\in\mathbb{R}}(1+|x|)^m\left|\frac{d^n}{dx^n}f(x)\right| < \infty.$$

The topology of $\mathcal{S}(\mathbb{R})$ is defined by the semi-norms $p_{m,n}$, $m,n=0,1,...$.

- *$\mathcal{S}_Q(\mathbb{R})$ the space of C^∞ functions f on $\mathbb{R}$ such that for all $m,n=0,1,...,$*

$$P_{m,n}(f) = p_{m,n}(Qf) < \infty.$$

The topology of $\mathcal{S}_Q(\mathbb{R})$ is defined by the semi-norms $P_{m,n}$, $m,n=0,1,...$.

- *$\mathcal{S}_{1/Q}(\mathbb{R})$ the space of C^∞ functions f on $\mathbb{R}$ such that for all $m,n=0,1,...,$*

$$N_{m,n}(f) = p_{m,n}(f/Q) < \infty.$$

The topology of $\mathcal{S}_{1/Q}(\mathbb{R})$ is defined by the semi-norms $N_{m,n}$, $m,n=0,1,...$.

- *$\mathcal{P}(\mathbb{R})$ the space of C^∞ functions f on $\mathbb{R}$ which are slowly increasing together with their derivatives; that is, for all $n=0,1,...,$ there is $m=0,1,...$ such that*

$$\sup_{x\in\mathbb{R}}(1+|x|)^{-m}\left|\frac{d^n}{dx^n}f(x)\right| < \infty.$$

- *$\mathcal{P}_{1/Q}(\mathbb{R})$ the space of C^∞ functions f on $\mathbb{R}$ such that $f/Q\in\mathcal{P}(\mathbb{R})$.*
- *$\mathcal{S}'(\mathbb{R})$ the space of tempered distributions on $\mathbb{R}$.*
- *$\mathcal{S}'_Q(\mathbb{R})$ the topological dual of $\mathcal{S}_Q(\mathbb{R})$.*
- *$\mathcal{S}'_{1/Q}(\mathbb{R})$ the topological dual of $\mathcal{S}_{1/Q}(\mathbb{R})$.*
- *$\mathcal{M}$ the map defined by*

$$\mathcal{M}f(x) = Q(x)f(x).$$

Remark 2.1. (*i*) It follows from (2.2) that $e_\gamma(i\lambda\cdot) \in \mathcal{P}(\mathbb{R})$ for $\lambda \in \mathbb{R}$.

(*ii*) $\mathcal{M}$ is a topological isomorphism

- from $\mathcal{S}_Q(\mathbb{R})$ onto $\mathcal{S}(\mathbb{R})$;
- from $\mathcal{S}(\mathbb{R})$ onto $\mathcal{S}_{1/Q}(\mathbb{R})$.

(*iii*) $\mathcal{M}$ is one-to-one from $\mathcal{P}(\mathbb{R})$ onto $\mathcal{P}_{1/Q}(\mathbb{R})$.

(*iv*) $\mathcal{M}^2$ is a topological isomorphism from $\mathcal{S}_Q(\mathbb{R})$ onto $\mathcal{S}_{1/Q}(\mathbb{R})$.

Lemma 2.2. *The Dunkl operator D_γ maps*

- $C^\infty(\mathbb{R})$ *into itself;*
- $\mathcal{P}(\mathbb{R})$ *into itself.*

Proof. (*i*) Let $f \in C^\infty(\mathbb{R})$. By writing

$$D_\gamma f(x) = f'(x) + (\gamma + \frac{1}{2}) \int_{-1}^{1} f'(tx)dt$$

and by using the derivation theorem under the integral sign we see that $D_\gamma f \in C^\infty(\mathbb{R})$ and

$$(D_\gamma f)^{(k)}(x) = f^{(k+1)}(x) + (\gamma + \frac{1}{2}) \int_{-1}^{1} f^{(k+1)}(tx)\, t^k dt \tag{2.3}$$

for all $k = 0, 1...$.

(*ii*) Let $f \in \mathcal{P}(\mathbb{R})$ and $n = 0, 1...$. By hypothesis there are $C > 0$ and $m = 0, 1...$ such that

$$\left| f^{(n+1)}(x) \right| \leq C(1 + |x|)^m,$$

for all $x \in \mathbb{R}$. So using (2.3) we obtain

$$\begin{aligned}
\left|(D_\gamma f)^{(n)}(x)\right| &\leq C(1+|x|)^m + C(2\gamma+1)\int_0^1 (1+t|x|)^m\, t^n dt \\
&\leq C(1+|x|)^m + C(2\gamma+1)(1+|x|)^m \int_0^1 t^n dt \\
&= C\left(1 + \frac{2\gamma+1}{n+1}\right)(1+|x|)^m \\
&\leq C(2\gamma+2)(1+|x|)^m,
\end{aligned}$$

for all $x \in \mathbb{R}$. This shows that $D_\gamma f \in \mathcal{P}(\mathbb{R})$. □

Lemma 2.3. *The Dunkl operator D_γ is a linear bounded operator from $\mathcal{S}(\mathbb{R})$ into itself.*

Proof. Let $f \in \mathcal{S}(\mathbb{R})$ and $m, n = 0, 1...$. By (2.3) we have for $|x| \leq 1$,

$$\begin{aligned}
(1+|x|)^m \left|(D_\gamma f)^{(n)}(x)\right| &\leq 2^m \left(1 + (\gamma + \frac{1}{2}) \int_{-1}^{1} |t|^n dt\right) \sup_{x\in[-1,1]} \left| f^{(n+1)}(x) \right| \\
&= 2^m \left(1 + \frac{2\gamma+1}{n+1}\right) \sup_{x\in[-1,1]} \left| f^{(n+1)}(x) \right| \\
&\leq 2^{m+1}(\gamma+1) p_{0,n+1}(f).
\end{aligned}$$

By (1.2) and Leibniz formula we have for $x \neq 0$,

$$(D_\gamma f)^{(n)}(x) = f^{(n+1)}(x) + (\gamma + \frac{1}{2})\, n! \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!} \frac{\Big(f^{(n-k)}(x) - (-1)^{n-k} f^{(n-k)}(-x)\Big)}{x^{k+1}}.$$

So for $|x| \geq 1$,

$$\begin{aligned} (1+|x|)^m \left|(D_\gamma f)^{(n)}(x)\right| &\leq (1+|x|)^m \left|f^{(n+1)}(x)\right| \\ &+ (\gamma + \frac{1}{2})\, n! \sum_{k=0}^{n} \frac{(1+|x|)^m \Big(\left|f^{(k)}(x)\right| + \left|f^{(k)}(-x)\right|\Big)}{k!} \\ &\leq p_{m,n+1}(f) + (2\gamma+1)\, n! \sum_{k=0}^{n} \frac{p_{m,k}(f)}{k!}. \end{aligned}$$

Therefore

$$p_{m,n}(D_\gamma f) \leq 2^{m+1}(\gamma+1) p_{0,n+1}(f) + p_{m,n+1}(f) + (2\gamma+1)\, n! \sum_{k=0}^{n} \frac{p_{m,k}(f)}{k!},$$

which shows that D_γ is bounded from $\mathcal{S}(\mathbb{R})$ into itself. □

Corollary 2.1. *(i) The differential-difference operator Λ is a linear bounded operator from $\mathcal{S}_{1/Q}(\mathbb{R})$ into itself.*

(ii) The dual operator of Λ, defined by

$$\widetilde{\Lambda} f(x) = f'(x) + (\gamma + \frac{1}{2}) \frac{f(x) - f(-x)}{x} - q(x) f(x),$$

is a linear bounded operator from $\mathcal{S}_Q(\mathbb{R})$ into itself.

Proof. From [[14], p. 7] we know that Λ and $\widetilde{\Lambda}$ are respectively linked to D_γ via the intertwining formulas

$$\Lambda \mathcal{M} f = \mathcal{M} D_\gamma f, \quad f \in C^\infty(\mathbb{R}), \tag{2.4}$$

$$D_\gamma \mathcal{M} f = \mathcal{M} \widetilde{\Lambda} f, \quad f \in C^\infty(\mathbb{R}). \tag{2.5}$$

Assertion (i) follows from (2.4), Remark 2.1(ii) and Lemma 2.3. Assertion (ii) follows from (2.5), Remark 2.1(ii) and Lemma 2.3. □

Remark 2.2. (i) A combination of (2.4) and (2.5) yields the formula

$$\Lambda \mathcal{M}^2 f = \mathcal{M}^2 \widetilde{\Lambda} f, \quad f \in C^\infty(\mathbb{R}). \tag{2.6}$$

(ii) By (2.4), Remark 2.1(iii) and Lemma 2.2 we see that Λ maps $\mathcal{P}_{1/Q}(\mathbb{R})$ into itself.

(iii) The duality between Λ and $\widetilde{\Lambda}$ is justified by the transposition relationship

$$\int_{\mathbb{R}} \Lambda f(x) g(x) |x|^{2\gamma+1} dx = -\int_{\mathbb{R}} f(x) \widetilde{\Lambda} g(x) |x|^{2\gamma+1} dx, \tag{2.7}$$

which is valid for any $f \in \mathcal{P}_{1/Q}(\mathbb{R})$ and $g \in \mathcal{S}_Q(\mathbb{R})$.

Notation 2.2. *Put*

$$m_\gamma = \frac{1}{2^{2\gamma+2}(\Gamma(\gamma+1))^2}.$$

We denote by

- $L^2_Q(\mathbb{R})$ *be the class of measurable functions* f *on* $\mathbb{R}$ *for which*

$$\|f\|_{2,Q} = \left(\int_{\mathbb{R}} |f(x)Q(x)|^2 \, |x|^{2\gamma+1} dx\right)^{1/2} < \infty.$$

- $L^2_{1/Q}(\mathbb{R})$ *be the class of measurable functions* f *on* $\mathbb{R}$ *for which*

$$\|f\|_{2,1/Q} = \left(\int_{\mathbb{R}} |f(x)/Q(x)|^2 \, |x|^{2\gamma+1} dx\right)^{1/2} < \infty.$$

The generalized Fourier transform of a function f in $\mathcal{S}_Q(\mathbb{R})$ is defined by

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}} f(x)Q(x)\, e_\gamma(-i\lambda x)|x|^{2\gamma+1} dx, \quad \lambda \in \mathbb{R}. \tag{2.8}$$

Remark 2.3. Recall that the one-dimensional Dunkl transform is defined for a function $f \in \mathcal{S}(\mathbb{R})$ by

$$\mathcal{F}_\gamma(f)(\lambda) = \int_{\mathbb{R}} f(x)\, e_\gamma(-i\lambda x)\, |x|^{2\gamma+1} dx, \quad \lambda \in \mathbb{R}. \tag{2.9}$$

By (2.8) and (2.9) observe that

$$\mathcal{F} = \mathcal{F}_\gamma \circ \mathcal{M}. \tag{2.10}$$

Theorem 2.1. *The generalized Fourier transform* $\mathcal{F}$ *is a topological isomorphism from* $\mathcal{S}_Q(\mathbb{R})$ *onto* $\mathcal{S}(\mathbb{R})$*. The inverse transform is given by*

$$\mathcal{F}^{-1}(g)(x) = \frac{1}{Q(x)} \int_{\mathbb{R}} g(\lambda)\, e_\gamma(i\lambda x)\, d\sigma(\lambda),$$

where

$$d\sigma(\lambda) = m_\gamma \, |\lambda|^{2\gamma+1} \, d\lambda.$$

Proof. It is well known that the one-dimensional Dunkl transform $\mathcal{F}_\gamma$ is a topological automorphism of $\mathcal{S}(\mathbb{R})$ and its inverse is given by

$$\mathcal{F}_\gamma^{-1}(g)(x) = \int_{\mathbb{R}} g(\lambda)\, e_\gamma(i\lambda x)\, d\sigma(\lambda).$$

The result follows now from (2.10) and Remark 2.1(ii). □

Theorem 2.2. (i) *For every* $f \in \mathcal{S}_Q(\mathbb{R})$ *we have the Plancherel formula*

$$\int_{\mathbb{R}} |f(x)|^2 \, (Q(x))^2 \, |x|^{2\gamma+1} dx = \int_{\mathbb{R}} |\mathcal{F}(f)(\lambda)|^2 \, d\sigma(\lambda).$$

(ii) *The generalized Fourier transform* $\mathcal{F}$ *extends uniquely to an isometric isomorphism from* $L^2_Q(\mathbb{R})$ *onto* $L^2(\mathbb{R}, \sigma)$.

The generalized Fourier transform of a distribution $S \in \mathcal{S}'_Q(\mathbb{R})$ is defined by

$$\langle \mathcal{F}(S), \psi \rangle = \langle S, \mathcal{F}^{-1}(\psi) \rangle, \quad \psi \in \mathcal{S}(\mathbb{R}).$$

Theorem 2.3. *The generalized Fourier transform $\mathcal{F}$ is one-to-one from $\mathcal{S}'_Q(\mathbb{R})$ onto $\mathcal{S}'(\mathbb{R})$.*

Notation 2.3. *(i) If $S \in \mathcal{S}'_Q(\mathbb{R})$ define $\Lambda S \in \mathcal{S}'_Q(\mathbb{R})$ by*

$$\langle \Lambda S, \psi\rangle = -\langle S, \widetilde{\Lambda}\psi\rangle, \quad \psi \in \mathcal{S}_Q(\mathbb{R}).$$

(ii) If $S \in \mathcal{S}'_{1/Q}(\mathbb{R})$ define $\widetilde{\Lambda} S \in \mathcal{S}'_{1/Q}(\mathbb{R})$ by

$$\langle \widetilde{\Lambda} S, \psi\rangle = -\langle S, \Lambda\psi\rangle, \quad \psi \in \mathcal{S}_{1/Q}(\mathbb{R}).$$

(iii) If $S \in \mathcal{S}'_{1/Q}(\mathbb{R})$ define $Q^2 S \in \mathcal{S}'_Q(\mathbb{R})$ by

$$\langle Q^2 S, \psi\rangle = \langle S, Q^2\psi\rangle, \quad \psi \in \mathcal{S}_Q(\mathbb{R}).$$

(iv) For $k = 0, 1, \ldots$ and $S \in \mathcal{S}'(\mathbb{R})$ let $x^k S \in \mathcal{S}'(\mathbb{R})$ be defined by

$$\langle x^k S, \psi\rangle = \langle S, x^k\psi\rangle, \quad \psi \in \mathcal{S}(\mathbb{R}).$$

(v) If f is a measurable function on $\mathbb{R}$, we write T_f for the functional

$$\langle T_f, \psi\rangle = \int_{\mathbb{R}} f(x)\psi(x)|x|^{2\gamma+1}dx.$$

Remark 2.4. From (2.6) we deduce the identity

$$\Lambda Q^2 S = Q^2 \widetilde{\Lambda} S, \quad S \in \mathcal{S}'_{1/Q}(\mathbb{R}). \tag{2.11}$$

Lemma 2.4. *Let $f \in L^2_Q(\mathbb{R})$ and $g(\lambda) = m_\gamma \mathcal{F}(f)(-\lambda)$. Then*

(i) $T_f \in \mathcal{S}'_{1/Q}(\mathbb{R})$;

(ii) $\mathcal{F}(T_{Q^2 f}) = T_g$.

Proof. (i) Let $\psi \in \mathcal{S}_{1/Q}(\mathbb{R})$ and m a positive integer such that $m > \gamma + 1$. By Schwarz inequality we have

$$|\langle T_f, \psi\rangle| = \left|\int_{\mathbb{R}} f(x)\psi(x)|x|^{2\gamma+1}dx\right| \leq \|f\|_{2,Q}\|\psi\|_{2,1/Q}.$$

But

$$\begin{aligned}
\|\psi\|_{2,1/Q} &= \left(\int_{\mathbb{R}} |\psi(x)/Q(x)|^2|x|^{2\gamma+1}dx\right)^{1/2} \\
&\leq p_{m,0}(\psi/Q)\left(\int_{\mathbb{R}} \frac{|x|^{2\gamma+1}}{(1+|x|)^{2m}}dx\right)^{1/2} \\
&\leq p_{m,0}(\psi/Q)\left(\int_{\mathbb{R}} \frac{dx}{(1+|x|)^{2m-2\gamma-1}}dx\right)^{1/2} \\
&= \frac{p_{m,0}(\psi/Q)}{\sqrt{m-\gamma-1}} \\
&= \frac{N_{m,0}(\psi)}{\sqrt{m-\gamma-1}},
\end{aligned}$$

which shows that T_f is bounded on $\mathcal{S}_{1/Q}(\mathbb{R})$.

(ii) Let $\psi \in \mathcal{S}(\mathbb{R})$. It is easily checked that

$$\overline{\mathcal{F}^{-1}(\psi)} = \mathcal{F}^{-1}(\widetilde{\psi}),$$

where $\widetilde{\psi}(\lambda)=\overline{\psi(-\lambda)}$. So using Theorem 2.2 we get

$$\begin{aligned}\langle\mathcal{F}\big(T_{Q^2f}\big),\psi\rangle &= \int_{\mathbb{R}} f(x)\mathcal{F}^{-1}(\psi)(x)(Q(x))^2|x|^{2\gamma+1}dx\\ &= \int_{\mathbb{R}} f(x)\,\overline{\mathcal{F}^{-1}(\widetilde{\psi})(x)}\,(Q(x))^2|x|^{2\gamma+1}dx\\ &= m_\gamma\int_{\mathbb{R}}\mathcal{F}(f)(\lambda)\psi(-\lambda)|\lambda|^{2\gamma+1}d\lambda\\ &= m_\gamma\int_{\mathbb{R}}\mathcal{F}(f)(-\lambda)\psi(\lambda)|\lambda|^{2\gamma+1}d\lambda\\ &= \langle T_g,\psi\rangle,\end{aligned}$$

which concludes the proof. □

Lemma 2.5. *Let $f\in\mathcal{S}_Q(\mathbb{R})$ and $S\in\mathcal{S}'_Q(\mathbb{R})$. Then for $k=1,2,...$ we have*

$$\mathcal{F}\big(\widetilde{\Lambda}^k f\big)(\lambda)=(i\lambda)^k\,\mathcal{F}(f)(\lambda), \tag{2.12}$$

$$\mathcal{F}\big(\Lambda^k S\big)=(-i\lambda)^k\,\mathcal{F}(S). \tag{2.13}$$

Proof. By (2.1), (2.4), (2.7) and Remark 2.1(i) we have

$$\begin{aligned}\mathcal{F}\big(\widetilde{\Lambda}^k f\big)(\lambda) &= \int_{\mathbb{R}} Q(x)e_\gamma(-i\lambda x)\,\widetilde{\Lambda}^k f(x)\,|x|^{2\gamma+1}dx\\ &= (-1)^k\int_{\mathbb{R}}\Lambda^k(Qe_\gamma(-i\lambda\cdot))(x)\,f(x)\,|x|^{2\gamma+1}dx\\ &= (-1)^k\int_{\mathbb{R}} Q(x)D_\gamma^k(e_\gamma(-i\lambda\cdot))(x)\,f(x)\,|x|^{2\gamma+1}dx\\ &= (i\lambda)^k\int_{\mathbb{R}} Q(x)e_\gamma(-i\lambda x)f(x)\,|x|^{2\gamma+1}dx\\ &= (i\lambda)^k\,\mathcal{F}(f)(\lambda).\end{aligned}$$

If $\psi\in\mathcal{S}(\mathbb{R})$ then

$$\langle\mathcal{F}\big(\Lambda^k S\big),\psi\rangle=\langle\Lambda^k S,\mathcal{F}^{-1}(\psi)\rangle=(-1)^k\langle S,\widetilde{\Lambda}^k\mathcal{F}^{-1}(\psi)\rangle.$$

But by (2.12),

$$\widetilde{\Lambda}^k\mathcal{F}^{-1}(\psi)=\mathcal{F}^{-1}\big((i\lambda)^k\psi\big).$$

So

$$\begin{aligned}\langle\mathcal{F}\big(\Lambda^k S\big),\psi\rangle &= (-1)^k\langle S,\mathcal{F}^{-1}\big((i\lambda)^k\psi\big)\rangle\\ &= (-1)^k\langle\mathcal{F}(S),(i\lambda)^k\psi\rangle\\ &= (-1)^k\langle(i\lambda)^k\mathcal{F}(S),\psi\rangle,\end{aligned}$$

which achieves the proof. □

Notation 2.4. *In all what follows assume $m=1,2,...$. Let $\mathcal{W}^m_{2,Q}$ be the Sobolev type space constructed by the differential-difference operator $\widetilde{\Lambda}$, i.e.,*

$$\mathcal{W}^m_{2,Q}=\Big\{f\in L^2_Q(\mathbb{R}):\widetilde{\Lambda}^j f\in L^2_Q(\mathbb{R}),\ j=1,2,...,m\Big\}.$$

More explicitly, $f\in\mathcal{W}^m_{2,Q}$ if and only if for each $j=1,2,...,m$, there is a function in $L^2_Q(\mathbb{R})$ abusively denoted by $\widetilde{\Lambda}^j f$, such that $\widetilde{\Lambda}^j T_f=T_{\widetilde{\Lambda}^j f}$.

Proposition 2.1. *For $f \in \mathcal{W}^m_{2,Q}$ we have*

$$\mathcal{F}\big(\widetilde{\Lambda}^m f\big)(\lambda) = (i\lambda)^m \mathcal{F}(f)(\lambda). \tag{2.14}$$

Proof. By the definition of $\mathcal{W}^m_{2,Q}$ we have

$$\widetilde{\Lambda}^m T_f = T_{\widetilde{\Lambda}^m f}\,.$$

It follows from (2.11), (2.13) and Lemma 2.4 that

$$\mathcal{F}\Big(Q^2\widetilde{\Lambda}^m T_f\Big) = \mathcal{F}\big(\Lambda^m Q^2 T_f\big) = \mathcal{F}\big(\Lambda^m T_{Q^2 f}\big) = (-i\lambda)^m\, \mathcal{F}\big(T_{Q^2 f}\big) = T_g,$$

with $g(\lambda) = m_\gamma\,(-i\lambda)^m\,\mathcal{F}(f)(-\lambda)$. Again by Lemma 2.4,

$$\mathcal{F}\Big(Q^2 T_{\widetilde{\Lambda}^m f}\Big) = \mathcal{F}\Big(T_{Q^2\widetilde{\Lambda}^m f}\Big) = T_h,$$

with $h(\lambda) = m_\gamma\,\mathcal{F}\big(\widetilde{\Lambda}^m f\big)(-\lambda)$. Identity (2.14) is now immediate. □

Recall that the Dunkl translation operators τ^x_γ, $x \in \mathbb{R}$ are defined by

$$\begin{aligned}\tau^x_\gamma f(y) &= \frac{1}{2}\int_{-1}^{1} f\left(\sqrt{x^2+y^2-2xyt}\right)\left(1+\frac{x-y}{\sqrt{x^2+y^2-2xyt}}\right)A_\gamma(t)dt\\ &+\frac{1}{2}\int_{-1}^{1} f\left(-\sqrt{x^2+y^2-2xyt}\right)\left(1-\frac{x-y}{\sqrt{x^2+y^2-2xyt}}\right)A_\gamma(t)dt,\end{aligned}$$

where

$$A_\gamma(t) = \frac{\Gamma(\gamma+1)}{\sqrt{\pi}\,\Gamma(\gamma+1/2)}(1+t)\left(1-t^2\right)^{\gamma-1/2}.$$

The generalized translation operators $\mathcal{T}^x$, $x \in \mathbb{R}$, tied to Λ are defined by

$$\mathcal{T}^x f(y) = Q(x)Q(y)\,\tau^x_\gamma(f/Q)(y).$$

The generalized dual translation operators are given by

$${}^t\mathcal{T}^x f(y) = \frac{Q(x)}{Q(y)}\,\tau^{-x}_\gamma(Qf)(y).$$

Proposition 2.2. *(i) Let $f \in L^2_{1/Q}(\mathbb{R})$. Then for all $x \in \mathbb{R}$, $\mathcal{T}^x f \in L^2_{1/Q}(\mathbb{R})$ and*

$$\|\mathcal{T}^x f\|_{2,1/Q} \le 2\,Q(x)\,\|f\|_{2,1/Q}.$$

(ii) Let $f \in L^2_Q(\mathbb{R})$. Then for all $x \in \mathbb{R}$, ${}^t\mathcal{T}^x f \in L^2_Q(\mathbb{R})$ and

$$\left\|{}^t\mathcal{T}^x f\right\|_{2,Q} \le 2\,Q(x)\,\|f\|_{2,Q}. \tag{2.15}$$

(iii) For $f \in L^2_Q(\mathbb{R})$ we have

$$\mathcal{F}\left({}^t\mathcal{T}^x f\right)(\lambda) = Q(x)e_\gamma(-i\lambda x)\,\mathcal{F}(f)(\lambda). \tag{2.16}$$

(iv) For $f \in L^2_{1/Q}(\mathbb{R})$ and $g \in L^2_Q(\mathbb{R})$ we have the duality relation

$$\int_{\mathbb{R}} \mathcal{T}^x f(y)g(y)|y|^{2\gamma+1}dy = \int_{\mathbb{R}} f(y)\,{}^t\mathcal{T}^x g(y)|y|^{2\gamma+1}dy.$$

3 Equivalence of K-Functionals and Modulus of Smoothness

Definition 3.1. Let $f \in L^2_Q(\mathbb{R})$ and $r > 0$. Then

(i) The generalized modulus of smoothness is defined by

$$\omega_m(f, r)_{2,Q} = \sup_{0<h\le r} \|\Delta_h^m f\|_{2,Q},$$

where

$$\Delta_h^m f = \left({}^t\mathcal{T}^h - Q(h)I\right)^m f,$$

I being the unit operator.

(ii) The generalized K-functional is defined by

$$K_m(f, r)_{2,Q} = \inf\left\{\|f - g\|_{2,Q} + r\left\|\widetilde{\Lambda}^m g\right\|_{2,Q} \,:\, g \in \mathcal{W}^m_{2,Q}\right\}.$$

We can now state the main result of this paper which establishes the equivalence between the generalized modulus of smoothness and the generalized K-functional.

Theorem 3.1. *There are two positive constants $c_1 = c_1(m, \gamma)$ and $c_2 = c_2(m, \gamma)$ such that for all $f \in L^2_Q(\mathbb{R})$ and $r > 0$:*

$$c_1 \left(Q(r)\right)^m K_m(f, r^m)_{2,Q} \le \omega_m(f, r)_{2,Q} \le c_2 \left(M(r)\right)^m K_m(f, r^m)_{2,Q}\,, \tag{3.1}$$

where $M(r) = \sup_{0<h\le r} Q(h)$.

Remark 3.1. If Q is increasing on $[0, \infty[$, then (3.1) may be written as

$$c_1\, \omega_m(f, r)_{2,Q} \le \left(Q(r)\right)^m K_m(f, r^m)_{2,Q} \le c_2\, \omega_m(f, r)_{2,Q}.$$

To simplify the proof of Theorem 3.1 we have to demonstrate first some preliminary results.

Lemma 3.1. *Let $f \in L^2_Q(\mathbb{R})$ and $h > 0$. Then*

$$\|\Delta_h^m f\|_{2,Q} \le 3^m (Q(h))^m \|f\|_{2,Q} \tag{3.2}$$

and

$$\mathcal{F}(\Delta_h^m f)(\lambda) = (Q(h))^m \left(e_\gamma(-i\lambda h) - 1\right)^m \mathcal{F}(f)(\lambda). \tag{3.3}$$

Proof. The result follows readily by using (2.15), (2.16) and an induction on m. □

Lemma 3.2. *For all $f \in \mathcal{W}^m_{2,Q}$ and $h > 0$ we have*

$$\|\Delta_h^m f\|_{2,Q} \le h^m (Q(h))^m \left\|\widetilde{\Lambda}^m f\right\|_{2,Q}. \tag{3.4}$$

Proof. By (2.14), (3.3), Lemma 2.1(v) and Theorem 2.2 we have

$$\begin{aligned}
\|\Delta_h^m f\|^2_{2,Q} &= \int_{\mathbb{R}} |\mathcal{F}(\Delta_h^m f)(\lambda)|^2 \, d\sigma(\lambda) \\
&= (Q(h))^{2m} \int_{\mathbb{R}} |1 - e_\gamma(-i\lambda h)|^{2m} \, |\mathcal{F}(f)(\lambda)|^2 \, d\sigma(\lambda) \\
&\le h^{2m} (Q(h))^{2m} \int_{\mathbb{R}} |\lambda|^{2m} \, |\mathcal{F}(f)(\lambda)|^2 \, d\sigma(\lambda) \\
&= h^{2m} (Q(h))^{2m} \int_{\mathbb{R}} \left|\mathcal{F}\big(\widetilde{\Lambda}^m f\big)(\lambda)\right|^2 d\sigma(\lambda) \\
&= h^{2m} (Q(h))^{2m} \left\|\widetilde{\Lambda}^m f\right\|^2_{2,Q},
\end{aligned}$$

which is the desired result. □

Notation 3.1. *For $f \in L^2_Q(\mathbb{R})$ and $\nu > 0$ define the function*

$$P_\nu(f)(x) = \frac{1}{Q(x)} \int_{-\nu}^{\nu} \mathcal{F}(f)(\lambda)\, e_\gamma(i\lambda x) d\sigma(\lambda).$$

Proposition 3.1. *Let $f \in L^2_Q(\mathbb{R})$ and $\nu > 0$. Then*

(*i*) *$P_\nu(f) \in C^\infty(\mathbb{R})$ and*

$$\widetilde{\Lambda}^k P_\nu(f)(x) = \frac{1}{Q(x)} \int_{-\nu}^{\nu} \mathcal{F}(f)(\lambda)\, (i\lambda)^k\, e_\gamma(i\lambda x) d\sigma(\lambda) \tag{3.5}$$

for all $k = 0, 1, \dots$.

(*ii*) *For all $k = 0, 1, \dots$, $\widetilde{\Lambda}^k P_\nu(f) \in L^2_Q(\mathbb{R})$ and*

$$\mathcal{F}\big(\widetilde{\Lambda}^k P_\nu(f)\big)(\lambda) = (i\lambda)^k\, \mathcal{F}(f)(\lambda)\, \chi_\nu(\lambda), \tag{3.6}$$

where

$$\chi_\nu(\lambda) = \begin{cases} 1 & \text{if } |\lambda| \le \nu, \\ 0 & \text{if } |\lambda| > \nu. \end{cases}$$

Proof. The fact that $P_\nu(f) \in C^\infty(\mathbb{R})$ follows from the derivation theorem under the integral sign. Identity (3.5) follows easily from (2.1) and (2.5). Assertion (*ii*) is a consequence of (3.5) and Theorem 2.2. □

Lemma 3.3. *There is a positive constant $c = c(\gamma)$ such that*

$$\|f - P_\nu(f)\|_{2,Q} \le c^{-m}\, (Q(1/\nu))^{-m}\, \|\Delta^m_{1/\nu} f\|_{2,Q}$$

for any $f \in L^2_Q(\mathbb{R})$ and $\nu > 0$.

Proof. By (3.6) and Theorem 2.2 we have

$$\begin{aligned} \|f - P_\nu(f)\|^2_{2,Q} &= \int_{\mathbb{R}} |1 - \chi_\nu(\lambda)|^2\, |\mathcal{F}(f)(\lambda)|^2\, d\sigma(\lambda) \\ &= \int_{|\lambda| \ge \nu} |\mathcal{F}(f)(\lambda)|^2\, d\sigma(\lambda). \end{aligned}$$

By Lemma 2.1(*iii*) there is a constant $c > 0$ which depends only on γ such that

$$|1 - e_\gamma(-i\lambda/\nu)| \ge c$$

for all $\lambda \in \mathbb{R}$ with $|\lambda| \ge \nu$. From this, (3.3) and Theorem 2.2 we get

$$\begin{aligned} \|f - P_\nu(f)\|^2_{2,Q} &\le c^{-2m} \int_{|\lambda| \ge \nu} |1 - e_\gamma(-i\lambda/\nu)|^{2m}\, |\mathcal{F}(f)(\lambda)|^2\, d\sigma(\lambda) \\ &= c^{-2m}\, (Q(1/\nu))^{-2m} \int_{|\lambda| \ge \nu} |\mathcal{F}(\Delta^m_{1/\nu} f)(\lambda)|^2\, d\sigma(\lambda) \\ &\le c^{-2m}\, (Q(1/\nu))^{-2m} \int_{\mathbb{R}} |\mathcal{F}(\Delta^m_{1/\nu} f)(\lambda)|^2\, d\sigma(\lambda) \\ &= c^{-2m}\, (Q(1/\nu))^{-2m} \|\Delta^m_{1/\nu} f\|^2_{2,Q}, \end{aligned}$$

which ends the proof. □

Corollary 3.1. *For all $f \in L^2_Q(\mathbb{R})$ and $\nu > 0$ we have*

$$\|f - P_\nu(f)\|_{2,Q} \leq c^{-m}\,(Q(1/\nu))^{-m}\,\omega_m(f, 1/\nu)_{2,Q},$$

where c is as in Lemma 3.3.

Lemma 3.4. *There is a positive constant $C = C(\gamma)$ such that*

$$\left\|\widetilde{\Lambda}^m P_\nu(f)\right\|_{2,Q} \leq (C\nu)^m (Q(1/\nu))^{-m}\,\|\Delta^m_{1/\nu} f\|_{2,Q}$$

for every $f \in L^2_Q(\mathbb{R})$ and $\nu > 0$.

Proof. By (3.6) and Theorem 2.2 we have

$$\begin{aligned}\left\|\widetilde{\Lambda}^m P_\nu(f)\right\|^2_{2,Q} &= \int_{-\nu}^{\nu} \lambda^{2m}\,|\mathcal{F}(f)(\lambda)|^2\,d\sigma(\lambda)\\ &= \int_{-\nu}^{\nu} \frac{\lambda^{2m}}{|1 - e_\gamma(-i\lambda/\nu)|^{2m}}\,|1 - e_\gamma(-i\lambda/\nu)|^{2m}\,|\mathcal{F}(f)(\lambda)|^2\,d\sigma(\lambda).\end{aligned}$$

Put

$$C = \sup_{|t|\leq 1} \frac{|t|}{|1 - e_\gamma(-it)|}.$$

By L'Hôpital's rule,

$$\lim_{t\to 0} \frac{|t|}{|1 - e_\gamma(-it)|} = 2\,(\gamma + 1).$$

This when combined with Lemma 2.1(iv) entails $0 < C < \infty$. Moreover,

$$\begin{aligned}\sup_{|\lambda|\leq\nu} \frac{\lambda^{2m}}{|1 - e_\gamma(-i\lambda/\nu)|^{2m}} &= \nu^{2m} \sup_{|\lambda|\leq\nu} \frac{(\lambda/\nu)^{2m}}{|1 - e_\gamma(-i\lambda/\nu)|^{2m}}\\ &= \nu^{2m} \sup_{|t|\leq 1} \frac{t^{2m}}{|1 - e_\gamma(-it)|^{2m}}\\ &= (C\,\nu)^{2m}.\end{aligned}$$

Therefore

$$\begin{aligned}\left\|\widetilde{\Lambda}^m P_\nu(f)\right\|^2_{2,Q} &\leq (C\,\nu)^{2m} \int_{-\nu}^{\nu} |1 - e_\gamma(-i\lambda/\nu)|^{2m}\,|\mathcal{F}(f)(\lambda)|^2\,d\sigma(\lambda)\\ &= (C\,\nu)^{2m}(Q(1/\nu))^{-2m} \int_{-\nu}^{\nu} |\mathcal{F}(\Delta^m_{1/\nu} f)(\lambda)|^2\,d\sigma(\lambda)\\ &\leq (C\,\nu)^{2m}(Q(1/\nu))^{-2m} \int_{\mathbb{R}} |\mathcal{F}(\Delta^m_{1/\nu} f)(\lambda)|^2\,d\sigma(\lambda)\\ &= (C\,\nu)^{2m}(Q(1/\nu))^{-2m}\,\|\Delta^m_{1/\upsilon} f\|^2_{2,Q},\end{aligned}$$

by virtue of (3.3) and Theorem 2.2. □

Corollary 3.2. *For any $f \in L^2_Q(\mathbb{R})$ and $\nu > 0$ we have*

$$\left\|\widetilde{\Lambda}^m P_\nu(f)\right\|_{2,Q} \leq (C\nu)^m (Q(1/\nu))^{-m}\,\omega_m(f, 1/\nu)_{2,Q},$$

where C is as in Lemma 3.4.

Proof of Theorem 3.1. (*i*) Let $h \in]0, r]$ and $g \in \mathcal{W}^m_{2,Q}$. By (3.2) and (3.4) we have

$$\begin{aligned}
\|\Delta^m_h f\|_{2,Q} &\leq \|\Delta^m_h (f-g)\|_{2,Q} + \|\Delta^m_h g\|_{2,Q} \\
&\leq 3^m (Q(h))^m \|f-g\|_{2,Q} + h^m (Q(h))^m \|\widetilde{\Lambda}^m g\|_{2,Q} \\
&\leq 3^m (M(r))^m \|f-g\|_{2,Q} + r^m (M(r))^m \|\widetilde{\Lambda}^m g\|_{2,Q} \\
&\leq 3^m (M(r))^m \left(\|f-g\|_{2,Q} + r^m \|\widetilde{\Lambda}^m g\|_{2,Q}\right).
\end{aligned}$$

Calculating the supremum with respect to $h \in]0, r]$ and the infimum with respect to all possible functions $g \in \mathcal{W}^m_{2,Q}$ we obtain

$$\omega_m(f, r)_{2,Q} \leq c_2\, (M(r))^m K_m\, (f, r^m)_{2,Q}\,,$$

with $c_2 = 3^m$.

(*ii*) Let ν be a positive real number. As $P_\nu(f) \in \mathcal{W}^m_{2,Q}$ it follows from the definition of the K-functional and Corollaries 3.1 and 3.2 that

$$\begin{aligned}
K_m\,(f, r^m)_{2,Q} &\leq \|f - P_\nu(f)\|_{2,Q} + r^m \|\widetilde{\Lambda}^m P_\nu(f)\|_{2,Q} \\
&\leq c^{-m}\,(Q(1/\nu))^{-m}\,\omega_m(f, 1/\nu)_{2,Q} + C^m r^m \nu^m (Q(1/\nu))^{-m} \omega_m(f, 1/\nu)_{2,Q} \\
&= (Q(1/\nu))^{-m}\left(c^{-m} + C^m (\nu r)^m\right) \omega_m(f, 1/\nu)_{2,Q}.
\end{aligned}$$

Since ν is arbitrary, by choosing $\nu = 1/r$ we get

$$c_1\,(Q(r))^m K_m(f, r^m)_{2,Q} \leq \omega_m(f, r)_{2,Q},$$

with $c_1 = (c^{-m} + C^m)^{-1}$. This completes the proof. □

4 Conclusion

We consider a singular differential-difference operator Λ on the real line which includes, as particular case, the Dunkl operator associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. By using an harmonic analysis corresponding to Λ, we construct generalized K-functionals and modulus of smoothness, which turn out to be equivalent.

Competing Interests

The authors declare that no competing interests exist.

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