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$* -$ Skew Polynomial Rings

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Abstract

Let $*$ be an involution over some ring. In this note $* -$ skew polynomial rings over commutative rings are studied along with $* -$ rigidity and $* -$ Armendariz property. Some interesting applications are demonstrated for uniserial rings.

Keywords: *$* -$ symmetry; $* -$ rigidity; $* -$ skew Armendariz property; $* -$ skew polynomial rings; uniserial rings.*

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1 Introduction

In the associative ring theory, symmetric rings were generalized and extended in various directions by several authors. In these generalizations and extensions rigidity and Armendariz property of rings play crucial roles. The aim of this note is to demonstrate some applications of $* -$ rigidity and $* -$ Armendariz property for $* -$ symmetric rings and $* -$ skew polynomial rings, where $*$ is an involution on the ring.

Lambek in [1] defined that a ring R with 1 is symmetric if for any elements $a,b,c \in R$, $abc = 0 \Rightarrow acb = 0$. For rings with 1, if $abc = 0 \Rightarrow acb = 0$ (or $bac = 0$,) then it also implies that all other remaining permutational products of these three elements are zero. Cohn in [2] defined that a ring R is reversible if for $a,b \in R$, $ab = 0 \Rightarrow ba = 0$. Extending these definitions to rings with involutions, it is defined that a ring R with involution $*$ is $* -$ symmetric if for any elements $a,b,c \in R$,

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$abc = 0 \Rightarrow acb^* = 0$, and that R is $*-$ reversible if for $a, b \in R$, $ab = 0 \Rightarrow ba^* = 0$ (see [3,4]). As in the case of $*-$ reversibility [4], there is no ambiguity between left and right $*-$ reversible rings, same is the case for left and right $*-$ symmetric rings. Quick calculations reveals that if for any elements $a, b, c \in R$, $abc = 0 \Rightarrow acb^* = 0$, then $b^*ac = 0$. Every $*-$ symmetric ring with 1 is symmetric, but the converse in general need not be true. For example [3; Example 1], for any prime p, consider the ring $(\mathbb{Z}_p \oplus \mathbb{Z}_p, +, \cdot)$ with component-wise addition and multiplication. Clearly $\mathbb{Z}_p \oplus \mathbb{Z}_p$ is symmetric and reversible but with the exchange involution $*$, it is neither $*-$ symmetric nor $*-$ reversible.

Note that these studies are not applicable for all classes of rings, as there are several classes of rings which do not adhere to an involution. For instance, the class of non-commutative generalized Klein- 4 rings $\mathbf{K}_{2^n}$ as studied in [5] and the class of the upper triangular matrix rings of the type $\begin{bmatrix} \mathbb{Z}_{2^k} & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ as discussed in ([6]; Example 1.5). In the case of commutative rings, every anti-automorphism is an automorphism, and an automorphism of degree 2 is an involution, so that identity map for commutative rings is always an involution.

In the study of rings with involutions rigidity plays important roles: *A ring R with the involution $*$ is called $*-$rigid, if for any $a \in R$, $aa^* = 0$, then $a = 0$.* Clearly, all domains, commutative or non-commutative, are $*-$ rigid, while $\mathbb{Z}_p \oplus \mathbb{Z}_p$ with the exchange involution is not $*-$ rigid.

In this paper our aim is to investigate properties of $*-$ skew polynomial rings along with $*-$ Armendariz property. These terms are defined in Section 2. Note that our definition of $*-$ skew polynomial rings forced the ring R to be commutative. So in Section 2 we have assumed that R is commutative. In Section 3 we picked a classic case of a factor polynomial ring of a $*-$ skew polynomial ring, say in the form, $R[x;*]/\langle x^n \rangle$, which in (upper triangular) matrix form is termed as Barnett matrix ring [7]. We showed that $R[x;*]$ and its factor ring $R[x;*]/\langle x^n \rangle$ are very opposite in nature (see Theorem 3.3). The uniserial property for an automorphism on R is also recalled.

Important note on notation: Note that if R is a ring that admits an involution $*$, then in the following the induced involution on a polynomial ring or a matrix ring will also be denoted by $*$. Thus the induced involution on the polynomial $\sum_{i=0}^{\alpha} a_i x^i$ $(\forall a_i \in R)$ in the polynomial ring $R[x]$ is defined by $\sum_{i=0}^{\alpha} a_i^* x^i$ $(\forall a_i \in R)$. Indeed, this is an involution on $R[x]$. Same are the cases for matrix rings and factor rings.

For definitions and terms from general ring theory [8] is a standard source. For related extensions and properties of symmetric and reversible rings we refer to [9-11] and the references therein. For terms related to $*-$reversible and $*-$ symmetric rings [3,4] may be referred.

The following results are proved in [3]

Lemma 1.1. *For a ring R with involution $*$ the following hold:*

(i) *If R is reduced and $*-$symmetric, then R is $*-$ reversible.*

(ii) *If R is $*$-reversible, then R is symmetric if and only if R* is $*$ -symmetric.
(iii) *R is $*$-rigid and $*-$ symmetric if and only if R is reduced and $*$-reversible.*
(iv) *R is $*$-rigid and semi-commutative if and only if R is semi prime and $*$-symmetric.*
(v) *If R is a $*$-rigid ring, then the following are equivalent:*

(1) *R is $*$-symmetric.*
(2) *R is symmetric.*
(3) *R is $*$-reversible.*
(4) *R is reversible.*

2 $*-$ skew Polynomial Rings

Throughout this section it is assumed that R is a commutative ring with the involution $*$ and with 1 such that $1^* = 1.$ Let x be an indeterminate which associate but do not commute with the elements of R. We define a $*-$ skew polynomial ring as follows:

Definition 2.1. *A $*-$ skew polynomial ring $R[x;*]$ of R is the ring consisting of all left polynomials of the form $\sum_{i=0}^{\alpha} a_i x^i$ $(\forall a_i \in R)$ with the multiplication defined by using the commutation formula $xa = a^* x$ for all $a \in R$.*

Remark: If we let R to be any ring, and follow the above commutation formula, we observe that for any elements $a, b \in R$,

$$x(ab) = (ab)^* x = (b^* a^*)x = b^*(ax) = b^*(xa) = (b^* x)a = (xb)a = x(ba).$$

This implies that R must be commutative.

The following example shows that there exists a $*-$ skew polynomial ring $R[x;*]$ over a commutative ring R which is neither commutative nor symmetric.

Example 2.2. Consider the ring $(\mathbb{Z}_2 \oplus \mathbb{Z}_2, +, \cdot),$ where $+$ and $\cdot$ are defined component-wise. This ring always adhere to the exchange involution $*$ defined via, $(a,b)^* = (b,a),\ \forall (a,b) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then $\mathbb{Z}_2 \oplus \mathbb{Z}_2[x;*]$ is a $*-$ skew polynomial ring with the commutation:

$$x(a,b) = (a,b)^* x = (b,a)x, \quad \forall (a,b) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

Now we see that in $\mathbb{Z}_2 \oplus \mathbb{Z}_2[x;*]$,

$$[(1,0)x^2 + (1,1)x + (0,1)][(0,1)x + (1,0)] = (0,0),$$

while

$$[(0,1)x+(1,0)][(1,0)x^2+(1,1)x+(0,1)]=(0.1)x^3+(1,1)x^2+(0,1)\neq 0.$$

So $Z_2 \oplus Z_2[x;*]$ is not commutative. Clearly, it is not reversible and because it has 1, it cannot be symmetric. Moreover, the $*-$ skew polynomial ring $Z_2 \oplus Z_2[x;*]$ is neither $*-$ symmetric nor $*-$ reversible ([4]; Proposition 5).

Note also one interesting fact that $Z_2 \oplus Z_2$ is reduced but $Z_2 \oplus Z_2[x;*]$ is not reduced, because $[(0,1)x]^2 = 0.$ Further, every $*-$ rigid ring is reduced [8], this shows that $Z_2 \oplus Z_2[x;*]$ is not $*-$ rigid as well. Hence we have a conclusion that if R is reduced then the skew polynomial ring $R[x;*]$ may not be reduced, or Armendariz.

In the following we show a deep link between $*-$ skew symmetric rings and $*-$ symmetric rings.

Theorem 2.3. *If $R[x;*]$ is symmetric then R is $*-$ symmetric. The converse holds if R is reduced.*

Proof: Assume that $a,b,c \in R,$ such that $abc=0.$ Then $abcx=0.$ If $R[x;*]$ is symmetric then,

$$xabc=(a^*x)bc=0 \Rightarrow b(a^*x)c=(ba^*)xc=(ba^*)c^*x=0$$
$$\Rightarrow (ba^*)c^*=0 \Rightarrow cab^*=0 \Rightarrow acb^*=0.$$

Hence R is $*-$ symmetric.

Conversely, let R be $*-$ symmetric and reduced. Consider the following triple product of polynomials in $R[x;*],$ and balance it to zero.

$$\left(\sum_{i=0}^{\alpha} a_i x^i\right)\left(\sum_{j=0}^{\beta} b_j x^j\right)\left(\sum_{k=0}^{\gamma} c_k x^k\right)=0. \tag{1}$$

We need to prove that $R[x;*]$ is symmetric. This means we will prove that

$$\left(\sum_{i=0}^{\alpha} a_i x^i\right)\left(\sum_{k=0}^{\gamma} c_k x^k\right)\left(\sum_{j=0}^{\beta} b_j x^j\right)=0. \tag{2}$$

By (1) the constant term is

$$a_0b_0c_0=0 \Rightarrow a_0c_0b_0=0. \tag{3}$$

The coefficients of x yield:

$$a_0b_0c_1+a_0b_1c_0^*+a_1b_0^*c_0^*=0. \tag{4}$$

$$\Rightarrow b_0c_0a_1b_0^*c_0^*=0 \Rightarrow a_1b_0c_0a_1b_0^*c_0^*=0.$$

Because R is $*-$ symmetric, so the symmetry and reducibility of R yields

$$a_1 b_0^* c_0^* = 0 \Rightarrow a_1 c_0^* b_0^* = 0.$$

Then,

$$a_0 b_0 c_1 + a_0 b_1 c_0^* = 0 \Rightarrow c_0 a_0 b_1 c_0^* = 0 \Rightarrow a_0 b_1 c_0 a_0 b_1 c_0^* = 0 \Rightarrow a_0 b_1 c_0^* = 0$$
$$\Rightarrow c_0 a_0 b_1 = 0 \Rightarrow a_0 c_0 b_1 = 0.$$

Again

$$a_0 b_0 c_1 + a_0 b_1 c_0^* = 0 \Rightarrow a_0 b_0 c_1 = 0 \Rightarrow a_0 c_1 b_0^* = 0.$$

Hence

$$a_0 c_0 b_1 + a_0 c_1 b_0^* + a_1 c_0^* b_0^* = 0. \tag{5}$$

Let us collect the coefficients of x^2 at one place:

$$a_0 b_0 c_2 + a_0 b_2 c_0 + a_2 b_0 c_0 + a_0 b_1 c_1^* + a_1 b_1^* c_0 + a_1 b_0^* c_1^* = 0. \tag{6}$$

We need to get:

$$a_0 c_0 b_2 + a_0 c_2 b_0 + a_2 c_0 b_0 + a_0 c_1 b_1^* + a_1 c_1^* b_0 + a_1 c_0^* b_1^* = 0. \tag{7}$$

This can be obtained by multiplying (6) by $a_0 b_1 c_1$, $a_1 b_1 c_0$, and $a_1 b_0 c_1$, consecutively, and apply the rules of $*-$ symmetric rings and reducibility, we get

$$a_0 c_1 b_1^* = a_1 c_1^* b_0 = a_1 c_0^* b_1^* = 0,$$

which leads to (7). The same processes will be continued until all terms are exhausted.

A ring R is called Armendariz, if for any pair of polynomials, $(f(x), g(x)) = \left(\sum_{i=0}^{\alpha} a_i x^i, \sum_{j=0}^{\beta} b_j x^j \right)$ in $R[x]$, $\forall a_i, b_j \in R$, where x is some commuting indeterminate, if $fg = 0$, then each product $a_i b_j = 0$.

For a non-commutative indeterminate, $\alpha-$ skew Armendariz rings were introduced in [9]. Following this concept, we introduce here $*-$ *skew-Armendariz* property for $*-$ skew symmetric rings. We need a modification in the original definition of the Armendariz property.

Definition 2.4. A ring R is called $*-$ *skew-Armendariz* if for any pair of polynomials, $\left(\sum_{i=0}^{\alpha} a_i x^i, \sum_{j=0}^{\beta} b_j x^j \right)$ in $R[x;*]$, $\forall a_i, b_j \in R$, where x is some indeterminate satisfying the commutation formula, $xa = a^* x, \forall a \in R$, whenever

$$\left(\sum_{i=0}^{\alpha} a_i x^i\right)\left(\sum_{j=0}^{\beta} b_j x^j\right) = 0,$$

then $\forall j = 0,1,2,3,\cdots,$

$$\left\{\begin{matrix} a_i b_j = 0 \text{ if } i = 0,2,4,\cdots \\ a_i b_j^* = 0 \text{ if } i = 1,3,5,\cdots \end{matrix}\right\}.$$

The following are outcomes of the Theorem 2.3 (see also [9,11]).

Lemma 2.5: *If R is $*-$rigid, then R is reduced and $*-$skew Armendariz.*

Proof: First notice that if R is $*-$rigid, then R becomes non-singular and hence reduced ([8]; Lemma 7.9 & Corollary 7.12).

Now let $(f(x), g(x)) = \left(\sum_{i=0}^{\alpha} a_i x^i, \sum_{j=0}^{\beta} b_j x^j\right)$ be a pair of polynomials in $R[x;*], \forall a_i, b_j \in R,$ such that $fg = 0.$ Then

(1) $a_0 b_0 = 0,$ and

(2) $a_0 b_1 + a_1 b_0^* = 0 \Rightarrow$

$$a_0 b_0 b_1 + a_1 b_0 b_0^* = 0 \Rightarrow a_1 b_0 a_1^* b_0^* = 0 \Rightarrow a_1 b_0^* = 0 \Rightarrow a_0 b_1 = 0.$$

(3) $a_0 b_2 + a_1 b_1^* + a_2 b_0 = 0 \Rightarrow$

$$b_0 b_0^* (a_0 b_2 + a_1 b_1^* + a_2 b_0) = 0 \Rightarrow b_0 b_0^* a_2 b_0 = 0 \Rightarrow (a_2 b_0)^2 ((a_2 b_0)^2)^* = 0 \Rightarrow a_2 b_0 = 0$$

$$a_0 b_2 + a_1 b_1^* = 0 \Rightarrow a_1 b_1 a_1^* b_1^* = 0 \Rightarrow a_1 b_1^* = 0 \Rightarrow a_0 b_2 = 0.$$

(4) $a_0 b_3 + a_1 b_2^* + a_2 b_1 + a_3 b_0^* = 0.$

Again we get,

$$b_0 b_0^* a_3 b_0^* = 0 \Rightarrow (a_3 b_0^*)^2 ((a_3 b_0^*)^2)^* = 0 \Rightarrow a_3 b_0^* = 0.$$

$$a_0 b_3 + a_1 b_2^* + a_2 b_1 = 0 \Rightarrow b_1 b_1^* a_2 b_1 = 0 \Rightarrow (a_2 b_1)^2 ((a_2 b_1)^2)^* = 0 \Rightarrow a_2 b_1 = 0.$$

Finally,

$$a_0 b_3 + a_1 b_2^* = 0 \Rightarrow (a_1 b_2^*)(a_1 b_2^*)^* = 0 \Rightarrow a_1 b_2^* = 0 \Rightarrow a_0 b_3 = 0.$$

We continue until we get all products in the form $a_i b_j = 0$ for even i, and $a_i b_j = 0$ for odd i. Which shows that R is $*-$ skew Armendariz.

Theorem 2.6. *The following are equivalent:*

(1) R *is* $*-$ *rigid;*
(2) $R[x;*]$ *is reduced;*
(3) R *is reduced and* $R[x;*]$ *is symmetric;*
(4) R *is reduced and* $*-$ *symmetric.*

Proof: (1) $\Longleftrightarrow$ (2) First assume that R is $*-$ rigid. Let, $f(x) = \sum_{i=0}^{\alpha} a_i x^i$ be a polynomial in $R[x;*]$ such that $f(x)^2 = 0$. Then because R is $*-$ Armendariz, for even subscripts we get $a_i^2 = 0 \Rightarrow a_0 = 0$, because R is reduced, and for odd subscripts, $a_i a_i^* = 0 \Rightarrow a_i = 0$. Hence $f(x) = 0$.

Conversely, if $R[x;*]$ is reduced, then being subring, R is reduced. Now let, for any $a \in R$, $aa^* = 0$. Then $(ax)^2 = 0 \Rightarrow ax = 0$. Hence $a = 0$.

$(2) \Rightarrow (3)$ Clearly every reduced ring is symmetric.
$(3) \Rightarrow (4)$ Holds from Theorem 2.3.
$(4) \Rightarrow (1)$ Let for any $a \in R$, $aa^* = 0 \Rightarrow aa = 0 \Rightarrow a = 0$.

Corollary 2.7. *Let a ring* R be $*-$ rigid and $f(x) = \sum_{i=0}^{\alpha} a_i x^i$, $g(x) = \sum_{j=0}^{\beta} b_j x^j \in R[x;*]$. Then $f(x)g(x) = 0$ if and only if $a_i b_j = 0$ for all $0 \le i \le \alpha$, $0 \le j \le \beta$.

Proof: Let $f(x)g(x) = 0$. Because R is $*-$ rigid, by Lemma 2.5, R is $*-$ skew Armendariz. So $a_i b_j = 0$ if i is even and $a_i b_j^* = 0$ if i is odd. Because R is also reduced, so it is reversible and by Lemma 1.1, it is $*-$ reversible. Hence $a_i b_j = 0$ for all $I = 1, \cdots, \alpha$. The converse is trivial.

Corollary 2.8. *If* R *is reduced and* $*-$ *skew - Armendariz, then* R *is* $*$ *symmetric. Hence* $R[x;*]$ *is symmetric*.

Corollary 2.9. *If* R *is a* $*-$ *skew - Armendariz ring, then* $R[x;*]$ *is symmetric if and only if* R *is* $*$ *symmetric.*

It is clear that R is non-singular if and only if $R[x]$ is non-singular. We prove the following.

Corollary 2.10. *Let* R *be* $*$ *-Armendariz.* R *is non-singular if and only if* $R[x;*]$ *is non-singular*.

Proof: It is already known that R is non-singular if and only if R is reduced. Being reduced and $*-$

Armendariz, R becomes $*-$ symmetric.

Now let $f(x)=\sum_{i=0}^{\alpha} a_i x^i \in R[x;*].$ Assume that

$$g(x)=\sum_{i=0}^{\beta} b_i x^i \in r_{R[x;*]}\langle f(x)\rangle \cap f(x)R[x;*],$$

where $r_{R[x;*]}\langle f(x)\rangle$ is the right annihilator of the ideal $\langle f(x)\rangle$ in $R[x;*].$ Then $f(x)g(x)=0$ and for some $h(x)=\sum_{i=0}^{\gamma} c_i x^i \in R[x;*],\ g(x)=f(x)h(x),$ which give

$$b_0=a_0c_0 \Rightarrow b_0^2=b_0a_0c_0 \Rightarrow 0 \Rightarrow b_0=0.$$

$$\begin{aligned} &b_1=a_0c_1+a_1c_0^* \Rightarrow b_1b_1=a_0b_1c_1+a_1b_1c_0^* \Rightarrow b_1b_1=a_1b_1c_0^* \Rightarrow \\ &\quad \Rightarrow b_1(b_1-a_1c_0^*)=0 \Rightarrow b_1^*(b_1-a_1c_0^*)=0 \Rightarrow b_1^*b_1=0 \Rightarrow b_1=0. \end{aligned}$$

$$b_2=a_0c_2+a_1c_1+a_2c_0^* \Rightarrow b_2^2=a_0b_2c_2+a_1b_2c_1+a_2b_2c_0^* \Rightarrow b_2(b_2-a_1c_1)=0.$$

By a similar argument as above, we conclude that $b_2=0.$ Continuing in this way, eventually, we will get that $g(x)=0.$ Hence the right annihilator ideal $r_{R[x;*]}\langle f(x)\rangle$ is non-singular. We conclude that $R[x;*]$ is right non-singular. Analogously, one can prove that it is left non-singular, hence non-singular.

Recall that an ideal I of a ring R is called $*-$ rigid, if for any $r\in R,\ rr^*\in I \Rightarrow r\in I.$ Also I is completely semiprime if and only if for any $r\in R,\ r^2\in I \Rightarrow r\in I.$

Proposition 2.11. *For a $*-$ ideal I of R the following statements are equivalent:*

(i) *I is a $*-$ rigid ideal.*
(ii) *I is completely semiprime, R/I is $*-$ Armendariz, and* $(R/I)[x;*]$ is reduced

Proof: $(i)\Rightarrow(ii)$ Let I be $*$ - rigid. If for any $r\in R,\ r^2\in I \Rightarrow (rr^*)(rr^*)^*\in I \Rightarrow$

$rr^*\in I \Rightarrow r\in I.$ Now let for any $a\in R,\ (a+I)(a^*+I)=0.$ Then $aa^*\in I \Rightarrow a\in I \Rightarrow R/I$ is $*-$ rigid. Hence by Lemma 2.5. R/I is reduced and $*-$ Armendariz.

$(ii)\Rightarrow(i)$ Assume that $aa^*\in I$ for some $a\in R.$ Then

$$(a+I)(a^*+I)=0 \Rightarrow (a+I)x(a+I)x=0 \Rightarrow a+I=0 \Rightarrow a\in I.$$

$*-$ skew Laurent polynomial rings can be defined analogously to that of $*-$ skew polynomial rings. Hence we have:

Proposition: 2.12. *Let R be a commutative ring with an involution $*$. Then $R[x;*]$ is $*-$skew Armendariz if and only if the $*-$skew Laurent polynomial ring $R[x,x^{-1};*]$ is $*-$skew Armendariz.*

Proof: Let $(f(x),g(x))=\left(\sum_{i=-s}^{\alpha}a_ix^i,\ \sum_{j=-t}^{\beta}b_jx^j\right)$ be a pair of polynomials in $R[x,x^{-1};*]$, $\forall a_i,b_j\in R$, such that $fg=0.$ Then $(x^sf(x),g(x)x^t)$ is a pair of polynomials in $R[x;*]$ such that

$$x^sf(x)g(x)x^t=0.$$

Because $R[x;*]$ is $*-$skew Armendariz which implies that either $a_ib_j^*=0,$ or $a_i^*b_j=0,$ or $a_ib_j=0,$ or $a_i^*b_j^*=0$ for all $-s\le i\le\alpha$ (with proper consideration of i to be even or odd) and $\forall j,$ such that $-t\le j\le\beta.$ Hence $R[x,x^{-1};*]$ is $*-$skew Armendariz. Converse is trivial.

3 Applications

3.1. First we discuss a general situation. Let R be a ring with 1, and not necessarily commutative. Let $R[x;\sigma]$ be the $\sigma-$ skew polynomial ring, in literature also termed as a twisted polynomial ring, where σ is an endomorphism on $R.$ Then $R[x;\sigma]/\langle x^n\rangle$ is a finite ring with a descending chain of principal ideals:

$$\langle x\rangle\supseteq\cdots\supseteq\langle x^{n-1}\rangle\supseteq\langle x^n\rangle=0.$$

This ring can also be written in the Barnett matrix form:

$$T_n(R,\sigma)=\left\{\left[a_{ij}\right]:\forall a_{ij}\in R\ :\ \begin{matrix}a_{ij}=0, & \forall i>j & i,j=1,2,\cdots,n\\ a_{ij}=a_{j-i}, & \forall i\le j & i,j=1,2,\cdots,n\end{matrix}\right\}\tag{3(a)}$$

Then,

$$R[x;\sigma]/\langle x^n\rangle\cong T_n(R;\sigma).$$

The isomorphism can be achieved in a compatible way by making a natural modification in the matrix multiplication. So, let $\left[a_{ij}\right],\ \left[b_{ij}\right]\in T_n(R,\sigma),$ with the rules of 3(a), and $\left[a_{ij}\right]\left[b_{jk}\right]=\left[c_{ik}\right],$ where

$$c_{ik}=\sum_{j=1}^{n}a_{ij}\sigma^{k-i}(b_{jk}).\tag{3(b)}$$

Naturally, in 3(b) rules of 3(a) apply.

3.2. Let $\sigma=1$ and t a commutative indeterminate. We write $T_n(R)$ in stead of $T_n(R,1).$ Thus, if R

is $*-$symmetric, then so is $R[t]/\langle t^n\rangle$. Hence for a commutative R with an involution $*$, the $*-$skew polynomial rings $R[t]/\langle t^n\rangle[x;*]$ and $T_n(R)[x;*]$ are symmetric. Several properties which are stated in Section 2, hold for R and extended rings in the form of factor polynomial rings and also in the form of Barnett matrix rings

$$T_n(R)[x;*] \mid\cong \frac{R[t]}{\langle t^n\rangle}[x;*].$$

Theorem 3.3: *Let R be commutative and self-injective:*

(a) *Then the following are equivalent:*

1. R *is von Neumann regular;*
2. R *is semi-hereditary;*
3. R *is Rickert;*
4. R *is Baer;*
5. R *is non-singular;*
6. R *is reduced.*

(b) *If R admits a rigid involution $*$, then the $*-$skew polynomial ring $R[x;*]$ satisfies all properties from (1) to (6) as listed in (a).*

(c) *The factor polynomial ring $R[x;*]/\langle x^n\rangle$ or the matrix ring $T_n(R,*)$ is not among any class of rings from (1) to (6) as listed in (a).*

Proof: (a) Follows from ([8]; (7.50, 7.52)).

(b) If R is self-injective, then so is $R[x;*]$. If R is $*-$rigid, then by Theorem 2.6., $R[x;*]$ is reduced, and so non-singular. The rest follows from ([8]; (7.50, 7.52)).

(c) Clearly, $R[x;*]/\langle x^n\rangle$ admits nilpotent elements so it is not reduced. The central elements of $R[x;*]/\langle x^n\rangle$ are of the form $x^2+\langle x^n\rangle,\ x^4+\langle x^n\rangle,\ x^2+x^4+\langle x^n\rangle,\ \cdots,$ etc., and because these are nilpotent elements, they become singular (see the details in ([8]; (7.11)). Hence $R[x;*]/\langle x^n\rangle$ is singular. Again by [8; (7.50)] $R[x;*]/\langle x^n\rangle$ is not among any class of rings from (1) to (6) as listed above.

3.4. Let F_q be a finite field of characteristic p, where q is some power of p. Let σ be an endomorphism on F_q, then one can always construct the $\sigma-$skew polynomial ring $\mathsf{F}_q[x;\sigma]$. If $\sigma\neq 1$, then $\mathsf{F}_q[x;\sigma]$ is non-commutative. Then the factor polynomial ring factored by $\langle x^n\rangle$ is

$$F = \frac{\mathbb{F}_q[x;\sigma]}{\langle x^n \rangle} \cong T_n(\mathbb{F}_q;\sigma),$$

for all $n \in \mathbb{N}$.

Clearly, $\mathbb{F}_q$ is $*-$ rigid for any involution $*$, it is non-singular, and hence $\mathbb{F}_q[x;*]$ is $*-$ rigid, reduced, non-singular, $*-$ symmetric, $*-$ reversible, etc., with $*$ considered as an induced involution on polynomials. Moreover $\mathbb{F}_q[x;*]$ satisfies all six equivalent conditions of Theorem 3.3 (a and b).

But by Theorem 3.3(c) F satisfies none of these six conditions of Theorem 3.3(a).

On the other hand $\mathbb{F}_q[x;\sigma]/\langle x^n \rangle$ or $T_n(\mathbb{F}_q;\sigma)$ has another interesting property as noticed in [12] and [13]:

Recall that a ring with 1 is right uniserial if it has a finite and unique composition series of right ideals. The ring is uniserial if it is both left and right uniserial.

Proposition 3.5. $T_n(\mathbb{F}_q;\sigma)$ *is a finite uniserial ring. Moreover, if* S *is a finite uniserial ring with the Jacobson radical* J, *then*

$$S \cong T_n(\mathbb{F}_q;\sigma),$$

for some automorphism σ *on* $\mathbb{F}_q$ *and* $q = |S/J|$.

Proof: The finite ring $T_n(\mathbb{F}_q;\sigma)$ is clearly uniserial as $\mathbb{F}_q[x;\sigma]/\langle x^n \rangle$ is uniserial because it has a unique chain of ideals

$$\langle x \rangle \supseteq \cdots \supseteq \langle x^{n-1} \rangle \supseteq \langle x^n \rangle = 0.$$

The rest holds from [13; Corollary 6] (see also [14; Theorem]).

Finally, Theorem 5.8 of [6] reveals that:

Corollary 3.6. $F = \mathbb{F}_q[x;\sigma]/\langle x^n \rangle$ *admits an anti-automorphism if and only if* σ *is an involution on* $\mathbb{F}_q$. *Moreover, if* σ *is an involution, then* F *admits an involution.*

4 Conclusion

This paper is in continuation of our investigations on $*$-reversible and $*$-symmetric rings published in [4] and [3], respectively. In this work we have studied $*$-skew polynomial rings and demonstrated some links with uniserial rings [13,14], and with the recent work of Wood on self-dual codes over non-commutative rings [6].

Competing Interests

Authors have declared that no competing interests exist.

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