British Journal of Mathematics & Computer Science
4(21): 3000-3013, 2014

SCIENCEDOMAIN *international*
www.sciencedomain.org

SCIENCEDOMAIN

Subordination Properties of p-valent Functions Defined by Linear Operators

R. M. EL-Ashwah[1*] **and M. E. Drbuk**[1]

[1] *Department of Mathematics, Faculty of Science, Damietta University, New Damietta 34517, Egypt.*

Original Research Article

Received: 18 June 2014
Accepted: 08 July 2014
Published: 10 August 2014

Abstract

In this paper we study different applications of the theory of differential subordination defined on the space of p-valent functions which are defined by linear operators. Also, some examples are given.

Keywords: Analytic functions, convex functions, linear operators, differential subordination and subordination.

2010 Mathematics Subject Classification: 30C45.

1 Introduction

Let $\mathcal{H}(U)$ be the class of analytic functions in the open unit disc U=$\{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{H}[a, p]$ be the subclass of $\mathcal{H}(\mathbb{U})$ consisting of functions of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + ... \qquad (a \in \mathbb{C};\ p \in \mathbb{N} = \{1, 2, ...\}).$$

Also, let $\mathcal{A}(p)$ be the subclass of the functions $f \in \mathcal{H}(\mathbb{U})$ of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N}), \tag{1.1}$$

and set $\mathcal{A} = \mathcal{A}(1)$ the class of univalent functions. Let $\mathcal{K}$ denotes the class of all convex functions in $\mathcal{A}$ which are satisfy the condition

$$\mathcal{K} = \left\{ f \in \mathcal{A} : \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \ (z \in \mathbb{U})\right\}.$$

*Corresponding author: E-mail: r_elashwah@yahoo.com

For $f, g \in \mathcal{H}(\mathrm{U})$, we say that f is subordinate to g, or g is superordinate to f, written as $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $\omega(z)$, which (by definition) is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1 \quad (z \in \mathrm{U})$ such that $f(z) = g(\omega(z)) \quad (z \in \mathrm{U})$. Furthermore, if the function g is univalent in U, then we have the following equivalence:

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

For $m \in \mathbb{Z}, \ell > -p, \lambda \geq 0$, Prajapat [1] introduced the operator
$J_p^m(\lambda, \ell) : \mathcal{A}(p) \longrightarrow \mathcal{A}(p)$, where

$$J_p^m(\lambda, \ell) = z^p + \sum_{n=p+1}^{\infty} \left(\frac{p + \ell + \lambda(n-p)}{p + \ell} \right)^m a_n z^n.$$

Also, let for $A > 0, a, c \in \mathbb{C}$ be such that $\Re(c - a) > 0$ and $\Re(a) > -Ap$, modified an Erdelyi-Kober type [2] integral operator, we define the linear operator
$\mathcal{I}_{p,A}^{a,c} : \mathcal{A}(p) \longrightarrow \mathcal{A}(p)$ by

$$\mathcal{I}_{p,A}^{a,c} f(z) = \frac{\Gamma(c + Ap)}{\Gamma(a + Ap)\Gamma(c - a)} \int_0^1 (1 - t)^{c-a-1} t^{a-1} f(zt^A) dt$$

$$= \frac{\Gamma(c + Ap)}{\Gamma(a + Ap)\Gamma(c - a)} \int_0^1 [(1 - t)^{c-a-1} t^{a+Ap-1} z^p + \sum_{n=p+1}^{\infty} (1 - t)^{c-a-1} t^{a+An-1} z^n] dt$$

$$= z^p + \frac{\Gamma(c + Ap)}{\Gamma(a + Ap)} \sum_{n=p+1}^{\infty} \frac{\Gamma(a + nA)}{\Gamma(c + nA)} a_n z^n \tag{1.2}$$

and

$$\mathcal{I}_{p,A}^{a,a} f(z) = f(z).$$

Let for $m \in \mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ and $A > 0, \lambda \geq 0, \ell > -p, a, c \in \mathbb{C}$ be such that $\Re(c - a) > 0$ and $\Re(a) > -Ap$, we define the linear operator $\mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A) : \mathcal{A}(p) \longrightarrow \mathcal{A}(p)$ by

$$\mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A) f(z) = (J_p^m(\lambda, \ell)(\mathcal{I}_{p,A}^{a,c} f(z)) = \mathcal{I}_{p,A}^{a,c}(J_p^m(\lambda, \ell) f(z))$$

$$\mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A) f(z) = z^p + \frac{\Gamma(c + Ap)}{\Gamma(a + Ap)} \sum_{n=p+1}^{\infty} \left(\frac{p + \ell + \lambda(n-p)}{p + \ell} \right)^m \frac{\Gamma(a + nA)}{\Gamma(c + nA)} a_n z^n. \tag{1.3}$$

It is readily verified from (1.3) that

$$\mathcal{J}_{\lambda,\ell}^{m,p}(a + 1, c, A) f(z) = \frac{a}{a + Ap} \mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A) f(z) + \frac{A}{a + Ap} z \left(\mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A) f(z) \right)' \tag{1.4}$$

and

$$\mathcal{J}_{\lambda,\ell}^{m+1,p}(a,c,A)f(z)=(1-\frac{p\lambda}{p+\ell})\mathcal{J}_{\lambda,\ell}^{m,p}(a,c,A)f(z)+\frac{\lambda}{p+\ell}z\left(\mathcal{J}_{\lambda,\ell}^{m,p}(a,c,A)f(z)\right)'. \tag{1.5}$$

Putting $c=a$ in (1.3) and by specializing the parameters $\lambda,\ \ell$ and p, we obtain the following operators studied by various authors:
(i) $\mathcal{J}_{\lambda,\ell}^{m,p}(a,a,A)f(z)=I_p^m(\lambda,\ell)f(z)$ $(\ell\geq 0,\ \lambda\geq 0,\ p\in\mathbb{N}$ and $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ (see [3]).
(ii) $\mathcal{J}_{1,\ell}^{m,p}(a,a,A)f(z)=I_p(m,\ell)f(z)$ $(\ell\geq 0, p\in\mathbb{N}$ and $m\in\mathbb{N}_0)$ (see [4,5]).
(iii) $\mathcal{J}_{\lambda,0}^{m,p}(a,a,A)=D_{\lambda,p}^m f(z)$ $(\lambda\geq 0, p\in\mathbb{N}$ and $m\in\mathbb{N}_0)$ (see [6]).
(iv) $\mathcal{J}_{1,0}^{m,p}(a,a,A)=D_p^m f(z)$ $(p\in\mathbb{N}$ and $m\in\mathbb{N}_0)$ (see [7,8]).
(v) $\mathcal{J}_{\lambda,\ell}^{-m,p}(a,a,A)=J_p^m(\lambda,\ell)f(z)$ $(\ell\geq 0,\ \lambda\geq 0, p\in\mathbb{N}$ and $m\in\mathbb{N}_0)$ (see [9,10,11]).
(vi) $\mathcal{J}_{1,1}^{-m,p}(a,a,A)=D^m f(z)$ $(m\in\mathbb{Z})$ (see [12]).
(vii) $\mathcal{J}_{1,\ell}^{m,1}(a,a,A)=I_\ell^m f(z)$ $(\ell\geq 0$ and $m\in\mathbb{N}_0)$ (see [13,14]).
(viii) $\mathcal{J}_{\lambda,0}^{m,1}(a,a,A)=D_\lambda^m f(z)$ $(\lambda\geq 0$ and $m\in\mathbb{N}_0)$ (see [15]).
(ix) $\mathcal{J}_{1,0}^{m,1}(a,a,A)=D^m f(z)$ $(m\in\mathbb{N}_0)$ (see [16]).
(x) $\mathcal{J}_{\lambda,1}^{-m,1}(a,a,A)=I_\lambda^{-m} f(z)$ $(\lambda\geq 0$ and $m\in\mathbb{N}_0)$ (see [17,18]).
(xi) $\mathcal{J}_{1,1}^{-m,1}(a,a,A)=I^m f(z)$ $(m\in\mathbb{N}_0)$ (see [19]).

Following definitions are due to Miller and Mocanu.

Definition 1.1 (20, Definition 2.2b, p. 21). . Denote by Q the class of functions f that are analytic and injective on $\overline{U}\backslash E(f)$,

$$E(f)=\left\{\zeta\in\partial U:\lim_{z\to\zeta}f(z)=\infty\right\},$$

and are such that $f'(\zeta)\neq 0$ for $\zeta\in\partial U\backslash E(f)$.

Definition 1.2 (20, p. 16). Let $\psi:\mathbb{C}^2\longrightarrow\mathbb{C}$ and let h be univalent in U. If $p(z)$ is analytic in U and satisfies the following first order differential subordination

$$\psi\left(p(z),zp'(z)\right)\prec h(z), \tag{1.6}$$

then $p(z)$ is called a solution of the differential subordination (1.6). A univalent function $q(z)$ is called a dominant of the solution of the differential subordination (1.6) or more simply, a dominant if $p(z)\prec q(z)$ for all $p(z)$ satisfying (1.6). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z)\prec q(z)$ for all dominants $q(z)$ of (1.6) is said to be the best dominant of (1.6).

A function $L(z,t):U\times[0,\infty)\longrightarrow\mathbb{C}$ is called a Löwner (subordination) chain if $L(.,t)$ is analytic and univalent in U for all $t\geq 0$, and $L(z,s)\prec L(z,t)$, $0\leq s\leq t$.

Recently, based on certain linear operators, some subordination preserving results have been obtained in [21], [22], [23], [24], [25], [26], [27] and [28]. In this paper, we obtain some subordination preserving properties associated with the new class of operators $\mathcal{J}_{\lambda,\ell}^{m,p}(a,c,A)$ involving complex parameters.

2 The Main Results

Lemma 2.1 (29, Theorem 1, p. 300). Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$, and let $h \in \mathcal{H}$(U) with $h(0) = c$. If $\Re(\beta h(z) + \gamma) > 0$ for $z \in U$, then the differential equation

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z), q(0) = c,$$

has an analytic solution in U, that satisfy $\Re\left(\beta q(z) + \gamma\right) > 0,\ z \in U.$

Lemma 2.2 [20, Theorem 2.3i, p. 35]. Suppose that the function $H : \mathbb{C}^2 \longrightarrow \mathbb{C}$ satisfies the condition

$$\Re\left(H(is, t))\right) \leq 0,$$

for all $s, t \in \mathbb{R}$ with $t \leq -n(1 + s^2)/2, n \in \mathbb{N}$. If the function $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + ...$ is analytic in U and

$$\Re\left\{H\left(p(z), zp'(z)\right)\right\} > 0 \quad (\text{z} \in U),$$

then $\Re(p(z)) > 0, z \in U.$ **Lemma 2.3** (20, Lemma 2.2d, p. 24). Let $q \in Q$ with $q(0) = a$, and let $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$ be analytic in U with $p(z) \neq a, n \in \mathbb{N}$. If p is not subordinate to q, then there exist the points $z_0 = r_0 e^{i\theta} \in U$ and $\zeta_0 \in \partial U \backslash E(f)$ such that $p(U_{r_0}) \subset q(U), p(z_0) = q(\zeta_0)$ and $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0), m \geq n$, where $U_{r_0} = \{z \in \mathbb{C} : |z| < r_0\}$.

Lemma 2.4 (30, Theorem 7, p. 882). Let $q \in \mathcal{H}$(U) and let $\phi : \mathbb{C}^2 \longrightarrow \mathbb{C}$, and set $\phi\left(q(z), zq'(z)\right) = h(z)$. If $L(z,t) = \phi\left(q(z), tzq'(z)\right)$ is a subordination chain and $p \in \mathcal{H}[a, 1] \cap Q$, then

$$h(z) \prec \phi\left(p(z), zp'(z)\right)$$

implies that

$$q(z) \prec p(z).$$

Furthermore, if the differential equation $\phi\left(q(z), tzq'(z)\right) = h(z)$ has a univalent solution $q \in Q$, then q is the best subordinant.

Lemma 2.5 [31, p. 159] Let $L(z,t) = a_1(t)z + a_2(t)z^2 + ...$, with $a_1(t) \neq 0$ for all $t \geq 0$ and $\lim\limits_{t \to +\infty} |a_1(t)| = +\infty$. Suppose that $L(., t)$ is analytic in U for all $t \geq 0, L(., t)$ is continuously differentiable on $[0, \infty)$ for all $z \in U$. If $L(z,t)$ satisfies

$$\Re\left(\frac{\partial L/\partial z}{\partial L/\partial t}\right) > 0 \qquad (t \geq 0; z \in U)$$

and

$$|L(z,t)| \leq K_0 \left|a_1(t)\right|, |z| < r_0 < 1, t \geq 0$$

for some positive constant K_0 and r_0, then $L(z,t)$ is a subordination chain.

Employing the techniques used in [32], we can prove the following theorem:

Theorem 2.1 *Let* for $\lambda > 0$,$m \in \mathbb{Z}, \ell > -p, p \in \mathbb{N}, A > 0, a, c \in \mathbb{C}$ satisfying $\Re(c - a) \geq 0$ and $\Re(a) > -Ap$, the operator $\mathcal{J}_{\lambda,\ell}^{m,p}(a, c, A)$ be defined by (1.3). Let for 0$\leq \beta \leq 1$,

$$\delta = \frac{(\ell + p)\,(a + Ap)}{(1 - \beta)\,p\lambda\,(a + Ap) + \beta Ap\,(\ell + p)} \tag{2.1}$$

be such that $\Re(\delta) \geq 1$ and for $g \in \mathcal{A}(p)$,

$$\varphi(z) = \frac{(1-\beta)\,\mathcal{J}_{\lambda,\ell}^{m+1,p}(a,c,A)g(z) + \beta\mathcal{J}_{\lambda,\ell}^{m,p}(a+1,c,A)g(z)}{z^{p-1}}, \tag{2.2}$$

satisfy

$$\Re\left(1+\frac{z\varphi''(z)}{\varphi'(z)}\right) > -\rho \quad (z \in U), \tag{2.3}$$

where $\rho = 0$ if $\Re(\delta) = 1$ and for $\Re(\delta) > 1$,

$$\rho \leq \begin{cases} \frac{\Re(\delta)-1}{2}, \Re(\delta) \leq 2, \\ \frac{1}{2(\Re(\delta)-1)}, \quad \Re(\delta) > 2, \end{cases} \tag{2.4}$$

and

$$(\Im(\delta))^2 \leq (\Re(\delta) - 1 - 2\rho)\left(\frac{1}{2\rho} - \Re(\delta) + 1\right), \tag{2.5}$$

the equality in (2.4) and (2.5) occur only when $\Im(\delta) = 0$. If $f \in \mathcal{A}(p)$ satisfies

$$\frac{(1-\beta)\,\mathcal{J}_{\lambda,\ell}^{m+1,p}(a,c,A)f(z)}{z^{p-1}} + \frac{\beta\mathcal{J}_{\lambda,\ell}^{m,p}(a+1,c,A)f(z)}{z^{p-1}} \prec \varphi(z) \quad (z \in U), \tag{2.6}$$

then

$$\frac{\mathcal{J}_{\lambda,\ell}^{m,p}(a,c,A)f(z)}{z^{p-1}} \prec \frac{\mathcal{J}_{\lambda,\ell}^{m,p}(a,c,A)g(z)}{z^{p-1}} \quad (z \in U). \tag{2.7}$$

Moreover, the function $\frac{\mathcal{J}_{\lambda,\ell}^{m,p}(a,c,A)g(z)}{z^{p-1}}$ is the best dominant of (2.6).

Proof. Let

$$F(z) = \frac{\mathcal{J}_{\lambda,\ell}^{m,p}(a,c,A)f(z)}{z^{p-1}} \text{ and } G(z) = \frac{\mathcal{J}_{\lambda,\ell}^{m,p}(a,c,A)g(z)}{z^{p-1}}. \tag{2.8}$$

By hypothesis, we first show that the function G is convex (univalent). For let

$$q(z) = 1 + \frac{zG''(z)}{G'(z)} \quad (z \in U). \tag{2.9}$$

Using (1.4) and (1.5) for $g \in \mathcal{A}(p)$, we have

$$\varphi(z) = \left(1 - \frac{1}{\delta}\right)G(z) + \frac{zG'(z)}{\delta}, \tag{2.10}$$

where δ is given by (1.2). On differentiating (2.10) and using (2.9), we have

$$1 + \frac{z\varphi''(z)}{\varphi'(z)} = q(z) + \frac{zq'(z)}{q(z)+\delta-1} =: h(z). \tag{2.11}$$

From (2.3) and (2.4), we have

$$\Re\left(h\left(z\right) + \delta - 1\right) > 0 \quad (z \in U),$$

and by Lemma 2.1, we deduce that the differential equation (2.11) has a solution $q \in \mathcal{H}(\mathbb{U})$, with $q(0) = h(0) = 1$.

Let

$$H(u,v) = u + \frac{v}{u+\delta-1} + \rho, \tag{2.12}$$

where ρ is given by (2.4).

From (2.3), (2.11) and (2.12)

$$\Re\left\{H\left(q(z), q'(z)\right)\right\} > 0 \quad (z \in U).$$

For all $s \in \mathbb{R}$ and $t \leq -\left(1+s^2\right)/2$, using (2.12), we have

$$\Re\left\{H\left(is,t\right)\right\} = \Re\left(is + \frac{t}{is+\delta-1} + \rho\right) \tag{2.13}$$

$$= \frac{(\Re\left(\delta\right)-1)t}{\left|is+\delta-1\right|^2} + \rho.$$

If $\Re\left(\delta\right)=1, \rho=0$, we have $\Re\left\{H\left(is,t\right)\right\}=0$ and if $\Re\left(\delta\right)>1$,

$$\Re\left\{H\left(is,t\right)\right\} \leq -\frac{\psi(s,\rho,\delta)}{2\left|is+\delta-1\right|^2}, \tag{2.14}$$

where

$$\psi(s,\rho,\delta) = \left(\Re\left(\delta\right)-1\right)\left(1+s^2\right) - 2\rho\left|is+\delta-1\right|^2$$

taking $\Re\left(\delta\right)-1=u$ and $\Im(\delta)=v$, we write

$$\psi(s,\rho,\delta) = (u-2\rho)s^2 - 4\rho v s + u - 2\rho(u^2+v^2).$$

If $v=0$, from (2.4), we have

$$\psi(s,\rho,\delta) = (u-2\rho)s^2 + (1-2\rho u)\,u \geq 0.$$

If $v \neq 0$, we assume that $u-2\rho>0$ for any $u>0$, we obtain

$$\psi(s,\rho,\delta) = (u-2\rho)\left(s-\frac{2\rho v}{u-2\rho}\right)^2 - \frac{4\rho^2 v^2}{u-2\rho} + u - 2\rho(u^2+v^2)$$

$$= (u-2\rho)\left(s-\frac{2\rho v}{u-2\rho}\right)^2 + u\left[1-2\rho\left(u+\frac{v^2}{u-2\rho}\right)\right] \geq 0,$$

from condition (2.5). Thus $\psi(s,\rho,\delta) \geq 0$ for all $s \in \mathbb{R}$. Hence, from (2.14) and (2.13), we have $\Re\left\{H\left(is,t\right)\right\} \leq 0$ for all $s \in \mathbb{R}$ and $t \leq -\left(1+s^2\right)/2$. Thus, by using Lemma 2.2, we conclude that $Re(q(z))>0$ for all z$\in U$, which proves that the function G defined by (2.8) is convex (univalent) in U.
We next prove that

$$F(z) \prec G(z) \quad (z \in U)\,, \tag{2.15}$$

if the subordination condition (2.6) holds. Without loss generality, we can assume G is analytic and univalent in $\overline{U}$and $G'(\zeta) \neq 0$ for $|\zeta|=1$. Otherwise, we replace F and G by $F_r(z)=F(rz)$ and $G_r(z)=G(rz)$, respectively, where $r \;\; (0<r<1)$. These functions satisfy the conditions of the theorem on $\overline{U}$, and we need to prove that $F_r(z) \prec G_r(z)$ for all $r \; (0<r<1)$, which enables us to prove (2.15) by letting $r \longrightarrow 1^-$.
Let us define a function $L(z,t)$ by

$$L(z,t) = \left(1-\frac{1}{\delta}\right)G(z) + \frac{(1+t)\,zG'(z)}{\delta} \quad (t \geq 0; z \in U). \tag{2.16}$$

Then,

$$\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = G'(0)\left(1+\frac{t}{\delta}\right) = 1+\frac{t}{\delta} \neq 0 (t \geq 0), \tag{2.17}$$

and this shows that the function $L(z,t)=a_1(t)z+a_2(t)z^2+...$, with $a_1(t)=1+\frac{t}{\delta} \neq 0$ for all $t \geq 0$ and $\lim\limits_{t\to+\infty}|a_1(t)|=+\infty$.
From (2.16) and using the assumption (2.1), for all $t \geq 0$, we have

$$\frac{|L(z,t)|}{|a_1(t)|} \leq \frac{|\delta-1|}{|\delta+t|}\,|G(z)| + \frac{(1+t)}{|\delta+t|}\left|zG'(z)\right|$$

$$\leq |G(z)| + \left|zG'(z)\right|. \tag{2.18}$$

Since the function G is convex and normalized in the unit disc, we have the following growth and distortion sharp bounds (see [33]).

$$\frac{r}{1+r} \leq |G(z)| \leq \frac{r}{1-r}, \quad |z| \leq r < 1,$$
$$\frac{1}{(1+r)^2} \leq \left|G'(z)\right| \leq \frac{1}{(1-r)^2}, \quad |z| \leq r < 1.$$

Using the upper bounds from these inequalities in (2.18), we have

$$\frac{|L(z,t)|}{|a_1(t)|} \leq \frac{r}{1-r} + \frac{1}{(1-r)^2} \leq \frac{1}{(1-r)^2}, \quad |z| \leq r < 1,\ t \geq 0$$

and thus, the assumptions of Lemma 2.5 hold. Furthermore, from (2.17), we have

$$\Re\left(\frac{\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right) = \Re(\delta) - 1 + (1+t)\Re\left(1 + \frac{zG''(z)}{G'(z)}\right) > 0 \quad (t \geq 0;\ z \in U)$$

and according to Lemma 2.5, the function $L(z,t)$ is a subordination chain. From the definition of subordination chain and definition of subordination, we obtain

$$L(\zeta,t) \notin L(U,0) = \varphi(U) \text{ whenever } \zeta \in \partial U, t \geq 0.$$

Suppose that F is not subordinate to G, then by Lemma 2.3 there exists points $z_0 \in U$ and $\zeta_0 \in \partial U$, and the number $t \geq 0$, such that

$$F(z_0) = G(\zeta_0) \text{ and } z_0 F'(z_0) = (1+t)\zeta_0 G'(\zeta_0).$$

From these two relations, and by virtue of the subordination condition (2.6), we deduce that

$$\begin{aligned} L(\zeta_0,t) &= \left(1 - \frac{1}{\delta}\right) G(\zeta_0) + \frac{(1+t)\,\zeta_0 G'(\zeta_0)}{\delta} \\ &= \left(1 - \frac{1}{\delta}\right) F(\zeta_0) + \frac{(1+t)\,\zeta_0 F'(\zeta_0)}{\delta} \\ &= \frac{1}{z^{p-1}}\left[(1-\beta)\,\mathcal{J}_{\lambda,\ell}^{m+1,p}(a,c,A)f(z_0) + \beta\mathcal{J}_{\lambda,\ell}^{m,p}(a+1,c,A)f(z_0)\right] \in \varphi(U) \end{aligned}$$

which contradicts the above observation that $L(\zeta,t) \notin \varphi(U)$. Therefore, the subordination condition (2.6) must imply the subordination given by (2.15). Considering $F(z) \prec G(z)$, we see that function G is best dominant, which completes the proof of theorem. ☐

Corollary 2.1 *Let* $\lambda > 0,\ m \in \mathbb{Z},\ \ell > -p,\ p \in \mathbb{N},\ 0 \leq \beta \leq 1,\ A > 0,\ a,\ c \in \mathbb{C}$ satisfying $\Re(c-a) \geq 0$ and $\Re(a) > -Ap$, $\Re(\delta)$ be given by (2.1). Let $g \in \mathcal{A}(p)$,

$$\begin{aligned} \psi_1(z) &= \frac{1}{z^{p-1}}\left[1 - \beta + \beta\frac{A(\ell+p)}{\lambda(a+Ap)}\right]\mathcal{J}_{\lambda,\ell}^{m+1,p}(a,c,A)g(z) \\ &\quad + \frac{\beta}{z^{p-1}}\left[1 - \frac{A(\ell+p)}{\lambda\,(a+Ap)}\right]\mathcal{J}_{\lambda,\ell}^{m,p}(a,c,A)g(z) \end{aligned} \tag{2.19}$$

satisfies

$$\Re\left(1 + \frac{z\psi_1''(z)}{\psi_1'(z)}\right) > -\rho \quad (z \in U),$$

where $\rho = 0$ if $\Re(\delta) = 1$ and for $\Re(\delta) > 1$, ρ is given by (2.4) with (2.5). If $f \in \mathcal{A}(p)$ and

$$\frac{1}{z^{p-1}}\left[1 - \beta + \beta\frac{A(\ell+p)}{\lambda(a+Ap)}\right]\mathcal{J}_{\lambda,\ell}^{m+1,p}(a,c,A)f(z)$$

$$+\frac{\beta}{z^{p-1}}\left[1-\frac{A(\ell+p)}{\lambda\,(a+Ap)}\right]\mathcal{J}^{m,p}_{\lambda,\ell}(a,c,A)f(z)\prec\psi_1(z),\quad z\in U \tag{2.20}$$

then

$$\frac{\mathcal{J}^{m,p}_{\lambda,\ell}(a,c,A)f(z)}{z^{p-1}}\prec\frac{\mathcal{J}^{m,p}_{\lambda,\ell}(a,c,A)g(z)}{z^{p-1}},\quad z\in U.$$

Moreover, the function $\frac{\mathcal{J}^{m,p}_{\lambda,\ell}(a,c,A)g(z)}{z^{p-1}}$ is the best dominant of (2.20).

Corollary 2.2 *Let* $\lambda>0,\ m\in\mathbb{Z},\ \ell>-p,\ p\in\mathbb{N},\ 0\le\beta\le1,\ A>0,\ a,\ c\in\mathbb{C}$ satisfying $\Re(c-a)\ge0$ and $\Re(a)>-Ap$, $\Re(\delta)$ be given by (2.1). Let $g\in\mathcal{A}(p)$,

$$\begin{aligned}\psi_2(z)\quad&=\quad\frac{1-\beta}{z^{p-1}}\left[1-\frac{\lambda(a+Ap)}{A(\ell+p)}\right]\mathcal{J}^{m,p}_{\lambda,\ell}(a,c,A)g(z)\\&\quad+\frac{1}{z^{p-1}}\left[(1-\beta)\,\frac{\lambda(a+Ap)}{A(\ell+p)}+\beta\right]\mathcal{J}^{m,p}_{\lambda,\ell}(a+1,c,A)g(z)\end{aligned}\tag{2.21}$$

satisfies

$$\Re\left(1+\frac{z\psi_2''(z)}{\psi_2'(z)}\right)>-\rho\quad(z\in U),$$

where $\rho=0$ if $\Re(\delta)=1$ and for $\Re(\delta)>1$, ρ is given by (2.4) with (2.5). Let $f\in\mathcal{A}(p)$ and

$$\frac{1-\beta}{z^{p-1}}\left[1-\frac{\lambda(a+Ap)}{A(\ell+p)}\right]\mathcal{J}^{m,p}_{\lambda,\ell}(a,c,A)f(z)$$

$$+\frac{1}{z^{p-1}}\left[(1-\beta)\,\frac{\lambda(a+Ap)}{A(\ell+p)}+\beta\right]\mathcal{J}^{m,p}_{\lambda,\ell}(a+1,c,A)f(z)\prec\psi_2(z),\quad z\in U \tag{2.22}$$

then

$$\frac{\mathcal{J}^{m,p}_{\lambda,\ell}(a,c,A)f(z)}{z^{p-1}}\prec\frac{\mathcal{J}^{m,p}_{\lambda,\ell}(a,c,A)g(z)}{z^{p-1}},\quad z\in U.$$

Moreover, the function $\frac{\mathcal{J}^{m,p}_{\lambda,\ell}(a,c,A)g(z)}{z^{p-1}}$ is the best dominant of (2.22).

Putting $\beta=0$ and 1, respectively, in Corollary 2.1 and 2.2, we obtain the following corollaries.

Corollary 2.3 Let $f,\ g\in\mathcal{A}(p),\ \ell>-p,\ p\in\mathbb{N},\ A>0,\ a,\ c\in\mathbb{C},\ \lambda>0$ be such that $\frac{\ell+p}{p\lambda}\ge1$, $m\in\mathbb{Z}$. Further, let

$$\Re\left(1+\frac{z\chi''(z)}{\chi'(z)}\right)>-\xi,\ z\in U,\ \chi(z)=\frac{\mathcal{J}^{m+1,p}_{\lambda,\ell}(a,c,A)g(z)}{z^{p-1}},$$

where $\xi=0$ if $\frac{\ell+p}{p\lambda}=1$ and for $\frac{\ell+p}{p\lambda}>1$,

$$\xi\le\begin{cases}\frac{\ell+p(1-\lambda)}{2p\lambda}<\frac{\ell+p}{p\lambda}\le2,\\ \frac{p\lambda}{2(\ell+p(1-\lambda))},\quad\frac{\ell+p}{p\lambda}>2.\end{cases}$$

Then $\frac{\mathcal{J}^{m+1,p}_{\lambda,\ell}(a,c,A)f(z)}{z^{p-1}}\prec\frac{\mathcal{J}^{m+1,p}_{\lambda,\ell}(a,c,A)g(z)}{z^{p-1}}\Rightarrow\frac{\mathcal{J}^{m,p}_{\lambda,\ell}(a,c,A)f(z)}{z^{p-1}}\prec\frac{\mathcal{J}^{m,p}_{\lambda,\ell}(a,c,A)g(z)}{z^{p-1}}$, $z\in U$. Moreover, the function $\frac{\mathcal{J}^{m,p}_{\lambda,\ell}(a,c,A)g(z)}{z^{p-1}}$ is the best dominant.

Corollary 2.4 *Let* $f,\ g\in\mathcal{A}(p),\ \lambda>0,\ m\in\mathbb{Z},\ \ell>-p,\ p\in\mathbb{N},\ A>0,\ a,\ c\in\mathbb{C}$ satisfying $\Re(c-a)\geq 0$ and $\Re(a)\geq 0$.Further, let

$$\Re\left(1+\frac{z\kappa''(z)}{\kappa'(z)}\right)>-\sigma,\ z\in U,\ \kappa(z)=\frac{\mathcal{J}_{\lambda,\ell}^{m,p}(a+1,c,A)g(z)}{z^{p-1}}$$

where $\sigma=0$ if $\Re(a)=0$ and for $\Re(a)>0$,

$$\sigma\leq\begin{cases}\frac{\Re(a)}{2Ap}, & \Re(a)\leq Ap,\\ \frac{Ap}{2\Re(a)}, & \Re(a)>Ap,\end{cases} \tag{2.23}$$

$$\left(\Im(a)\right)^2\leq\left(\Re(a)-2\sigma Ap\right)\left(\frac{Ap}{2\sigma}-\Re(a)\right) \tag{2.24}$$

equality in (2.23) and (2.24) occur only if $\Im(a)=0$. Then $\frac{\mathcal{J}_{\lambda,\ell}^{m,p}(a+1,c,A)f(z)}{z^{p-1}}\prec\frac{\mathcal{J}_{\lambda,\ell}^{m,p}(a+1,c,A)g(z)}{z^{p-1}}$ $\Rightarrow\frac{\mathcal{J}_{\lambda,\ell}^{m,p}(a,c,A)f(z)}{z^{p-1}}\prec\frac{\mathcal{J}_{\lambda,\ell}^{m,p}(a,c,A)g(z)}{z^{p-1}},\ z\in U$. Moreover, the function $\frac{\mathcal{J}_{\lambda,\ell}^{m,p}(a,c,A)g(z)}{z^{p-1}}$ is the best dominant.

Remark 2.1. Putting $p=1$ in the our main results, we obtain the results obtained by Raina and Sharma [32].

Applications

We will give an interesting special case of our main results, obtained for an appropriate choice of the function g and the corresponding parameters.
Thus, for $\lambda>0,\ A>0\ ,a,\ c\in\mathbb{C}$ satisfying $\Re(c-a)\geq 0$, $\Re(a)>-A$, and $\beta<1$, let consider the function $g\in\mathcal{A}$ defined by

$$g(z)=z+\sum_{n=1}^{\infty}a_{n+1}z^{n+1}\quad(z\in U),$$

with

$$\begin{aligned}a_{n+1}\ =\ &\tfrac{1}{n+1}\left(\tfrac{\ell+1}{1+\ell+\lambda n}\right)^m\tfrac{\Gamma(a+A)}{\Gamma(c+A)}\tfrac{\Gamma(c+(n+1)A)}{\Gamma(a+(n+1)A)}\\ &.\left(1+n\left[(1-\beta)\tfrac{\lambda}{1+\ell}+\beta\tfrac{A}{a+A}\right]\right)^{-1}\binom{-2(\rho+1)}{n}\quad(n\geq 1)\,,\end{aligned} \tag{2.25}$$

where ρ is given by (2.4)

$$\binom{\theta}{n}=\frac{\theta(\theta-1)...(\theta-n+1)}{n!},\ \theta\in\mathbb{C},\ n\in\mathbb{N}. \tag{2.26}$$

If the function φ is given by (2.2), with $p=1$, then

$$\varphi(z)=\frac{1-(1+z)^{-(2\rho+1)}}{2\rho+1}\quad(z\in U),$$

where the power is principal one, i.e.

$$(1+z)^{-(2\rho+1)}\Big|_{z=0}=1.$$

We can see that

$$\Re\left(1+\frac{z\varphi''(z)}{\varphi'(z)}\right)=\Re\frac{1-(2\rho+1)\,z}{1+z}>-\rho\quad(z\in U)\,,$$

and from Theorem 2.1, we can obtain:

Example 2.1 Let $0 \leq \beta \leq 1$, $\lambda > 0$, $A > 0$, a , $c \in \mathbb{C}$ satisfying $\Re(c-a) \geq 0$, $\Re(a) > -A$ and let ρ is given by (2.4).

If $f \in \mathcal{A}$ such that

$$(1-\beta)\,\mathcal{J}_{\lambda,\ell}^{m+1,1}(a,c,A)f(z) + \beta\mathcal{J}_{\lambda,\ell}^{m,1}(a+1,c,A)f(z) \prec \tfrac{1-(1+z)^{-(2\rho+1)}}{2\rho+1},$$

then

$$\mathcal{J}_{\lambda,\ell}^{m,1}(a,c,A)f(z) \prec z + \sum_{n=1}^{\infty} \tfrac{1}{n+1}\left(1+n\left[(1-\beta)\tfrac{\lambda}{1+\ell}+\beta\tfrac{A}{a+A}\right]\right)^{-1}\tbinom{-2(\rho+1)}{n} z^{n+1},$$

and the right-hand side function is the best subordinant.

Also, let consider the function $g \in \mathcal{A}$ defined by

$$g(z) = z + \sum_{n=1}^{\infty} a_{n+1} z^{n+1} \quad (z \in U),$$

with a_{n+1} is given by (2.25), ρ is given by (2.4) and $\binom{\theta}{n}$ is given by (2.26).
If the function ψ_1 is given by (2.19), with $p=1$, then

$$\psi_1(z) = \frac{1-(1+z)^{-(2\rho+1)}}{2\rho+1} \quad (z \in U),$$

where the power is principal one, i.e.

$$(1+z)^{-(2\rho+1)}\Big|_{z=0} = 1.$$

We can see that

$$\Re\left(1+\frac{z\psi_1''(z)}{\psi_1'(z)}\right) = \Re\frac{1-(2\rho+1)\,z}{1+z} > -\rho \quad (z \in U),$$

and from Corollary 2.1, we can obtain:

Example 2.2 *Let* $\lambda > 0$, $m \in \mathbb{Z}$, $\ell > -1$, $0 \leq \beta \leq 1$, $A > 0$, a , $c \in \mathbb{C}$ satisfying $\Re(c-a) \geq 0$ and $\Re(a) > -A$, $\Re(\delta)$ be given (2.1). If $f \in \mathcal{A}$ such that

$$\left[1-\beta+\beta\tfrac{A(\ell+1)}{\lambda(a+A)}\right]\mathcal{J}_{\lambda,\ell}^{m+1,1}(a,c,A)f(z) + \beta\left[1-\frac{A(\ell+1)}{\lambda\,(a+A)}\right]\mathcal{J}_{\lambda,\ell}^{m,1}(a,c,A)f(z)$$
$$\prec \quad \tfrac{1-(1+z)^{-(2\rho+1)}}{2\rho+1},$$

then

$$\mathcal{J}_{\lambda,\ell}^{m,1}(a,c,A)f(z) \prec z + \sum_{n=1}^{\infty} \tfrac{1}{n+1}\left(1+n\left[(1-\beta)\tfrac{\lambda}{1+\ell}+\beta\tfrac{A}{a+A}\right]\right)^{-1}\tbinom{-2(\rho_1+1)}{n} z^{n+1},$$

and the right-hand side function is the best dominant.

Finally, let consider the function $g \in \mathcal{A}$ defined by

$$g(z) = z + \sum_{n=1}^{\infty} a_{n+1} z^{n+1} \quad (z \in U),$$

with a_{n+1} is given by (2.25), ρ is given by (2.4) and $\binom{\theta}{n}$ is given by (2.26).

If the function $\psi_2(z)$ is given by (2.21), with $p = 1$, then

$$\psi_2(z) = \frac{1-(1+z)^{-(2\rho+1)}}{2\rho+1} \qquad (z \in U),$$

where the power is principal one, i.e.

$$(1+z)^{-(2\rho+1)}\Big|_{z=0} = 1.$$

We can see that

$$\Re\left(1+\frac{z\psi_2''(z)}{\psi_2'(z)}\right) = \Re\frac{1-(2\rho+1)\,z}{1+z} > -\rho \quad (z \in U)\,,$$

and from Corollary 2.2, we can obtain:

Example 2.3 *Let* $\lambda > 0$, $m \in \mathbb{Z}$, $\ell > -1$, $0 \leq \beta \leq 1$, $A > 0$, $a, c \in \mathbb{C}$ satisfying $\Re(c-a) \geq 0$ and $\Re(a) > -A$, $\Re(\delta)$ be given (2.1). If $f \in \mathcal{A}$ such that

$$(1-\beta)\left[1-\tfrac{\lambda(a+A)}{A(\ell+1)}\right]\mathcal{J}_{\lambda,\ell}^{m,1}(a,c,A)f(z) + \left[(1-\beta)\tfrac{\lambda(a+A)}{A(\ell+1)}+\beta\right]\mathcal{J}_{\lambda,\ell}^{m,1}(a+1,c,A)f(z)$$
$$\prec \quad \tfrac{1-(1+z)^{-(2\rho+1)}}{2\rho+1},$$

then

$$\mathcal{J}_{\lambda,\ell}^{m,1}(a,c,A)f(z) \prec z + \sum_{n=1}^{\infty}\tfrac{1}{n+1}\left(1+n\left[(1-\beta)\tfrac{\lambda}{1+\ell}+\beta\tfrac{A}{a+A}\right]\right)^{-1}\tbinom{-2(\rho_1+1)}{n}z^{n+1},$$

and the right-hand side function is the best dominant.

3 Conclusions

In this work, analytic p-valent functions defined on the unit disc, are studied with help of new transformation. This transformation is the modified an Erdelyi-Kober type [2] integral operator combining with Prajapat operator [1]. Using the new transformation and the techniques of differential subordination we obtained subordination theorems. Many interesting particular cases of main thm are emphasized in the form of corollaries and examples.

Acknowledgment

The authors thank the referees for their valuable suggestions which led to improvement of this paper.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Prajapat JK. Subordination and super ordination preserving properties for generalized multiplier transformation operator. Math. Comput. Modelling. 2012;55:1456-1465.

[2] Kiryakova V. Generalized Fractional Calculus and Applications, Pitman Research Notes in Mathematics Series, 301, John Willey & Sons, Inc. New York; 1994.

[3] Catas A. On certain classes of p-valent functions defined by multiplier tranformations, in Proc. Book of the International Symposium on Geometric Functions Theory and Applications, Istanbul, Turkey. 2007:241-250.

[4] Kumar SS, Taneja HC, Ravichandran V. Classes multivalent functions defined by Dziok-Srivastava linear operaor and multiplier transformations, Kyungpook Math. J. 2006;46:97-109.

[5] Srivastava HM, Suchithra K, Adolf Stephen B, Sivasubramanian S. Inclusion and neighborhood properties of certian subclasses of multivalent functions of complex order. J. Ineq. Pure Appl. Math. 2006;7(5):191, 1-8.

[6] Aouf MK, El-Ashwah RM, El-Deeb SM. Some inequalities for certain p-valent functions involving an extended multiplier transformations, Proc. Pakistan Acad. Sci. 2009;46(4):217-221.

[7] Aouf MK, Mostafa AO. On a subclass of n-p-valent prestarlike functions, Comput. Math. Appl. 2008;55:851-861.

[8] Kamali M, Orhan H. On a subclass of certian starlike functions with negative coefficients, Bull. Korean Math. Soc. 2004;41(1):53-71.

[9] Aouf MK, Mostafa AO, El-Ashwah RM. Sandwich theorems for p-valent functions defined by a certain integral operator, Math. Comput. Modelling. 2011;53:1647-1653.

[10] El-Ashwah RM, Aouf MK. Some properties of new integral Operator. Acta Univ. Apulensis. 2010;24:51-61.

[11] Srivastava HM, Aouf MK, El-Ashwah RM. Some inclusion relationships associated with a certain class of integral operators, Asian European J. Math. 2010;3(4):667-684.

[12] Patel J, Sahoo P. Certain subclasses of multivalent analytic Functions. Indian J. Pure Appl. Math. 2003;34:487-500.

[13] Cho NE, Srivastava HM. Argument estimates of certain analytic functions defined by a class of multiplier transformations. Math. Comput. Modelling. 2003;37(1-2):39-49.

[14] Cho NE, Kim TH. Multiplier transformations and strongly close-to-convex functions, Bull. Korean Math. Soc. 2003;40(3):399-410.

[15] Al-Oboudi FM. On univalent functions defined by a generalized Salagean operator, Indian J. Math. Math. Sci. 2004;25-28: 1429-1436.

[16] Salagean GS. Subclasses of univalent functions, Lecture Notes in Math. (Springer-Verlag). 1983;1013:362-372.

[17] Patel J. Inclusion relations and convolution properties of certain subclasses of analytic functions defined by generalized Salagean operator, Bull. Belg. Math. Soc. Simon Stevin. 2008;15:33-47.

[18] Aouf MK, Seoudy TM. On differential sandwich theorems of analytic functions defined by generalized Salagean integral operator. Appl. Math. Letter. 2011;24:1364-1368.

[19] Flett TM. The dual of an identity of Hardy and Little wood and some related inequalities. J. Math. Anal. Appl. 1972;38:746-765.

[20] Miller SS, Mocanu PT. Differential Subordinations. Theory and Applications, Marcel Dekker, New York and Basel; 2000.

[21] Ali RM, Ravichandran V, Seenivasagan N. Differential subordination and superordination of analytic functions defined by the multiplier transformation. Math. Inequal. Appl. 2009;12:123-139.

[22] Ali RM, Ravichandran V, Seenivasagan N. Subordination and superordination on Schwarzian derivatives. J. Inequal. Appl. 2008; Art. ID 712328, 18.

[23] Bulboaca T. Integral operators that preserve the subordination, Bull. Korean Math. Soc. 1997;34: 627-636.

[24] Cho NE, Owa S. Double subordination-preserving properties for certain integral operators. J. Inequal. Appl. 2007; Art. ID 83073, 10 pp.

[25] Cho NE, Srivastava HM. A class of nonlinear integral operators preserving subordination and superordination, Integral Transforms Spec. Funct. 2007;18(1-2):95-107.

[26] Miller SS, Mocanu PT, Reade MO. Subordination-preserving integral operators, Trans. Amer. Math. Soc. 1984;283:605-615.

[27] Shanmugam TN, Sivasubramanian S, Srivastava HM. Differential sandwich theorems for certain subclasses of analytic functions involving multiplier transformations, Integral Transforms Spec. Funct. 2006;17(12):889-899.

[28] Wang ZG, Xiang RG, Darus M. A family of integral operators preserving subordination and superordination, Bull. Malays. Math. Sci. Soc. 2010;33(2):121-131.

[29] Miller SS, Mocanu PT. Univalent solutions of Briot-Bouquet differential equations. J. Differential Eqns. 1985;56(3):297-309.

[30] Miller SS, Mocanu PT. Subordinants of differential super ordinations, Complex Var. Theory Appl. 2003;48:815-826.

[31] Pommerenke Ch. Univalent functions, Vanderhoeck and Ruprecht, Göttingen; 1975.

[32] Raina RK. Sharma P. Subordination preserving properties associated with a class of operators, LXVIII. 2013;217-228.

[33] Gronwall TH. Some remarks on conformal representation, Ann. Math. 1914;16/15:72-76.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
www.sciencedomain.org/review-history.php?iid=623&id=6&aid=5700