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A note on binomial transform of the generalized fifth order Jacobsthal numbers

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Abstract: In this paper, we define the binomial transform of the generalized fifth order Jacobsthal sequence and as special cases, the binomial transform of the fifth order Jacobsthal, fifth order Jacobsthal-Lucas, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas sequences will be introduced. We investigate their properties in details.

Keywords: Binomial transform; Fifth order Jacobsthal sequence; Fifth order Jacobsthal numbers; Binomial transform of fifth order Jacobsthal sequence; Binomial transform of fifth order Jacobsthal-Lucas sequence.

MSC: 11B37; 11B39; 11B83.

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1. Introduction and preliminaries

In this paper, we introduce the binomial transform of the generalized fifth order Jacobsthal sequence and we investigate, in detail, two special cases which we call them the binomial transform of the fifth order Jacobsthal and fifth order Jacobsthal-Lucas sequences. We investigate their properties in the next sections. In this section, we present some properties of the generalized (r,s,t,u,v) sequence (generalized Pentanacci) sequence.

The generalized (r,s,t,u,v) sequence (the generalized Pentanacci sequence or 5-step Fibonacci sequence)

$$\{W_n\}_{n\geq 0} = \{W_n(W_0,W_1,W_2,W_3,W_4;r,s,t,u,v)\}_{n\geq 0}$$

is defined by the fifth-order recurrence relations

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5}, \qquad W_0 = a, W_1 = b, W_2 = c, W_3 = d, W_4 = e\,, \tag{1}$$

where the initial values W_0, W_1, W_2, W_3, W_4 are arbitrary complex (or real) numbers and r,s,t,u,v are real numbers. Pentanacci sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [1–5]. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{u}{v}W_{-(n-1)} - \frac{t}{v}W_{-(n-2)} - \frac{s}{v}W_{-(n-3)} - \frac{r}{v}W_{-(n-4)} + \frac{1}{v}W_{-(n-5)}$$

for $n = 1,2,3,....$ Therefore, recurrence (1) holds for all integer n.

As $\{W_n\}$ is a fifth order recurrence sequence (difference equation), it's characteristic equation is

$$x^5 - rx^4 - sx^3 - tx^2 - ux - v = 0\,, \tag{2}$$

whose roots are $\alpha,\beta,\gamma,\delta,\lambda$. Note that we have the following identities:

$$\begin{aligned}
\alpha+\beta+\gamma+\delta+\lambda &= r,\\
\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\alpha\delta+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta &= -s,\\
\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\gamma+\alpha\gamma\delta+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta &= t,\\
\alpha\beta\lambda\gamma+\alpha\beta\lambda\delta+\alpha\beta\gamma\delta+\alpha\lambda\gamma\delta+\beta\lambda\gamma\delta &= -u\\
\alpha\beta\gamma\delta\lambda &= v.
\end{aligned}$$

Generalized Pentanacci numbers can be expressed, for all integers n, using Binet's formula.

Theorem 1. *(Binet's formula of generalized (r,s,t,u,v) numbers (generalized Pentanacci numbers))*

$$\begin{aligned}
W_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)}\\
&\quad+\frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)}+\frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}+\frac{p_5\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}, \qquad (3)
\end{aligned}$$

where

$$\begin{aligned}
p_1&=W_4-(\beta+\gamma+\delta+\lambda)W_3+(\beta\lambda+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta)W_2-(\beta\lambda\gamma+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta)W_1+(\beta\lambda\gamma\delta)W_0,\\
p_2&=W_4-(\alpha+\gamma+\delta+\lambda)W_3+(\alpha\lambda+\alpha\gamma+\alpha\delta+\lambda\gamma+\lambda\delta+\gamma\delta)W_2-(\alpha\lambda\gamma+\alpha\lambda\delta+\alpha\gamma\delta+\lambda\gamma\delta)W_1+(\alpha\lambda\gamma\delta)W_0,\\
p_3&=W_4-(\alpha+\beta+\delta+\lambda)W_3+(\alpha\beta+\alpha\lambda+\beta\lambda+\alpha\delta+\beta\delta+\lambda\delta)W_2-(\alpha\beta\lambda+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\delta)W_1+(\alpha\beta\lambda\delta)W_0,\\
p_4&=W_4-(\alpha+\beta+\gamma+\lambda)W_3+(\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\beta\gamma+\lambda\gamma)W_2-(\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\beta\lambda\gamma)W_1+(\alpha\beta\lambda\gamma)W_0,\\
p_5&=W_4-(\alpha+\beta+\gamma+\delta)W_3+(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)W_2-(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)W_1+(\alpha\beta\gamma\delta)W_0.
\end{aligned}$$

Usually, it is customary to choose r,s,t,u,v so that the Eq. (2) has at least one real (say α) solutions. Eq. (3) can be written in the following form:

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4\delta^n + A_5\lambda^n,$$

where

$$\begin{aligned}
A_1 &= \frac{p_1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)},\\
A_2 &= \frac{p_2}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)},\\
A_3 &= \frac{p_3}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)},\\
A_4 &= \frac{p_4}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)},\\
A_5 &= \frac{p_5}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}.
\end{aligned}$$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence W_n.

Lemma 1. *Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized (r,s,t,u,v) sequence $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n x^n$ is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0+(W_1-rW_0)x+(W_2-rW_1-sW_0)x^2+(W_3-rW_2-sW_1-tW_0)x^3+(W_4-rW_3-sW_2-tW_1-uW_0)x^4}{1-rx-sx^2-tx^3-ux^4-vx^5}. \qquad (4)$$

We next find Binet formula of generalized (r,s,t,u,v) numbers $\{W_n\}$ by the use of generating function for W_n.

Theorem 2. *(Binet's formula of generalized (r,s,t,u,v) numbers)*

$$\begin{aligned} W_n &= \frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ &\quad +\frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)}+\frac{q_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}+\frac{q_5\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}, \end{aligned} \tag{5}$$

where

$$\begin{aligned} q_1&=W_0\alpha^4+(W_1-rW_0)\alpha^3+(W_2-rW_1-sW_0)\alpha^2+(W_3-rW_2-sW_1-tW_0)\alpha+(W_4-rW_3-sW_2-tW_1-vW_0), \\ q_2&=W_0\beta^4+(W_1-rW_0)\beta^3+(W_2-rW_1-sW_0)\beta^2+(W_3-rW_2-sW_1-tW_0)\beta+(W_4-rW_3-sW_2-tW_1-vW_0), \\ q_3&=W_0\gamma^4+(W_1-rW_0)\gamma^3+(W_2-rW_1-sW_0)\gamma^2+(W_3-rW_2-sW_1-tW_0)\gamma+(W_4-rW_3-sW_2-tW_1-vW_0), \\ q_4&=W_0\delta^4+(W_1-rW_0)\delta^3+(W_2-rW_1-sW_0)\delta^2+(W_3-rW_2-sW_1-tW_0)\delta+(W_4-rW_3-s3W_2-tW_1-vW_0), \\ q_5&=W_0\lambda^4+(W_1-rW_0)\lambda^3+(W_2-rW_1-sW_0)\lambda^2+(W_3-rW_2-sW_1-tW_0)\lambda+(W_4-rW_3-sW_2-tW_1-vW_0). \end{aligned}$$

Matrix formulation of W_n can be given as [6]:

$$\begin{pmatrix} W_{n+4} \\ W_{n+3} \\ W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t & u & v \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_4 \\ W_3 \\ W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{6}$$

In fact, Kalman give the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \\ W_{n+3} \\ W_{n+4} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ r & s & t & u & v \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \\ W_4 \end{pmatrix}.$$

Next, we consider two special cases of the generalized (r,s,t,u,v) sequence $\{W_n\}$ which we call them (r,s,t,u,v) and Lucas (r,s,t,u,v) sequences. (r,s,t,u,v) sequence$\{G_n\}_{n\geq 0}$ and Lucas (r,s,t,u,v) sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the fifth-order recurrence relations

$$\begin{aligned} G_{n+5} &= rG_{n+4}+sG_{n+3}+tG_{n+2}+uG_{n+1}+vG_n, & (7) \\ G_0 &= 0, G_1=1, G_2=r, G_3=r^2+s, G_4=r^3+2sr+t, \\ H_{n+5} &= rH_{n+4}+sH_{n+3}+tH_{n+2}+uH_{n+1}+vH_n, & (8) \\ H_0 &= 5, H_1=r, H_2=2s+r^2, H_3=r^3+3sr+3t, H_4=r^4+4r^2s+4tr+2s^2+4u. \end{aligned}$$

The sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{u}{v}G_{-(n-1)}-\frac{t}{v}G_{-(n-2)}-\frac{s}{v}G_{-(n-3)}-\frac{r}{v}G_{-(n-4)}+\frac{1}{v}G_{-(n-5)}, \\ H_{-n} &= -\frac{u}{v}H_{-(n-1)}-\frac{t}{v}H_{-(n-2)}-\frac{s}{v}H_{-(n-3)}-\frac{r}{v}H_{-(n-4)}+\frac{1}{v}H_{-(n-5)}, \end{aligned}$$

for $n=1,2,3,\ldots$ respectively. Therefore, recurrences (7) and (8) hold for all integers n.

For more details on the generalized (r,s,t,u,v) numbers, see [4].

Some special cases of (r,s,t,u,v) sequence $\{G_n(0,1,r,r^2+s,r^3+2sr+t;r,s,t,u,v)\}$ and Lucas (r,s,t,u,v) sequence $\{H_n(5,r,2s+r^2,r^3+3sr+3t,r^4+4r^2s+4tr+2s^2+4u;r,s,t,u,v)\}$ are as follows:

1. $G_n(0,1,1,2,4;1,1,1,1,1) = P_n$, Pentanacci sequence,
2. $H_n(5,1,3,7,15;1,1,1,1,1) = Q_n$, Pentanacci-Lucas sequence,
3. $G_n(0,1,2,5,13;2,1,1,1,1) = P_n$, fifth-order Pell sequence,
4. $H_n(5,2,6,17,46;2,1,1,1,1) = Q_n$, fifth-order Pell-Lucas sequence.

For all integers n, (r,s,t,u,v) and Lucas (r,s,t,u,v) numbers (using initial conditions in (3) or (5)) can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= \frac{\alpha^{n+3}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{\beta^{n+3}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} + \frac{\gamma^{n+3}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} \\ &\quad + \frac{\delta^{n+3}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} + \frac{\lambda^{n+3}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}, \\ H_n &= \alpha^n + \beta^n + \gamma^n + \delta^n + \lambda^n, \end{aligned}$$

respectively.

Lemma 1 gives the following results as particular examples (generating functions of (r,s,t,u,v), Lucas (r,s,t,u,v) and modified (r,s,t,u,v) numbers).

Corollary 1. *Generating functions of (r,s,t,u,v), Lucas (r,s,t,u,v) and modified (r,s,t,u,v) numbers are*

$$\begin{aligned} \sum_{n=0}^{\infty} G_n x^n &= \frac{x}{1-rx-sx^2-tx^3-ux^4-vx^5}, \\ \sum_{n=0}^{\infty} H_n x^n &= \frac{5-4rx-3sx^2-2tx^3-ux^4}{1-rx-sx^2-tx^3-ux^4-vx^5}, \end{aligned}$$

respectively.

The following theorem shows that the generalized Pentanacci sequence W_n at negative indices can be expressed by the sequence itself at positive indices.

Theorem 3. *For $n \in \mathbb{Z}$, for the generalized Pentanacci sequence (or generalized (r,s,t,u,v)-sequence or 5-step Fibonacci sequence) we have the following:*

$$\begin{aligned} W_{-n} &= \frac{1}{24} v^{-n} (W_0 H_n^4 - 4W_n H_n^3 + 3W_0 H_{2n}^2 + 12H_n^2 W_{2n} - 6W_0 H_n^2 H_{2n} - 6W_0 H_{4n} - 8W_n H_{3n} - 12H_{2n} W_{2n} \\ &\quad -24H_n W_{3n} + 24W_{4n} + 8W_0 H_n H_{3n} + 12W_n H_n H_{2n}) \\ &= v^{-n}(W_{4n} - H_n W_{3n} + \frac{1}{2}(H_n^2 - H_{2n})W_{2n} - \frac{1}{6}(H_n^3 + 2H_{3n} - 3H_{2n}H_n)W_n \\ &\quad + \frac{1}{24}(H_n^4 + 3H_{2n}^2 - 6H_n^2 H_{2n} - 6H_{4n} + 8H_{3n}H_n)W_0). \end{aligned}$$

Proof. For the proof, see [5], Theorem 1. □

Using Theorem 3, we have the following corollary, see [5], Corollary 4.

Corollary 2. *For $n \in \mathbb{Z}$, we have*

$$H_{-n} = \frac{1}{24} v^{-n} (H_n^4 + 3H_{2n}^2 - 6H_n^2 H_{2n} - 6H_{4n} + 8H_{3n} H_n).$$

Note that G_{-n} and H_{-n} can be given as follows by using $G_0 = 0$ and $H_0 = 5$ in Theorem 3:

$$\begin{aligned} G_{-n} &= v^{-n}(G_{4n} - H_n G_{3n} + \frac{1}{2}(H_n^2 - H_{2n})G_{2n} - \frac{1}{6}(H_n^3 + 2H_{3n} - 3H_{2n}H_n)G_n), \\ H_{-n} &= \frac{1}{24} v^{-n}(H_n^4 + 3H_{2n}^2 - 6H_n^2 H_{2n} - 6H_{4n} + 8H_{3n}H_n), \end{aligned}$$

respectively.

Table 1. A few generalized fifth order Jacobsthal numbers

n	V_n	V_{-n}
0	V_0	...
1	V_1	$\frac{1}{2}V_4-\frac{1}{2}V_1-\frac{1}{2}V_2-\frac{1}{2}V_3-\frac{1}{2}V_0$
2	V_2	$\frac{3}{4}V_3-\frac{1}{4}V_1-\frac{1}{4}V_2-\frac{1}{4}V_0-\frac{1}{4}V_4$
3	V_3	$\frac{7}{8}V_2-\frac{1}{8}V_1-\frac{1}{8}V_0-\frac{1}{8}V_3-\frac{1}{8}V_4$
4	V_4	$\frac{15}{16}V_1-\frac{1}{16}V_0-\frac{1}{16}V_2-\frac{1}{16}V_3-\frac{1}{16}V_4$
5	$2V_0+V_1+V_2+V_3+V_4$	$\frac{31}{32}V_0-\frac{1}{32}V_1-\frac{1}{32}V_2-\frac{1}{32}V_3-\frac{1}{32}V_4$
6	$2V_0+3V_1+2V_2+2V_3+2V_4$	$\frac{31}{64}V_4-\frac{33}{64}V_1-\frac{33}{64}V_2-\frac{33}{64}V_3-\frac{33}{64}V_0$
7	$4V_0+4V_1+5V_2+4V_3+4V_4$	$\frac{95}{128}V_3-\frac{33}{128}V_1-\frac{33}{128}V_2-\frac{33}{128}V_0-\frac{33}{128}V_4$

Next, we consider the case $r=1, s=1, t=1, u=1, v=2$ and in this case we write $V_n=W_n$. A generalized fifth order Jacobsthal sequence $\{V_n\}_{n\geq 0}=\{V_n(V_0,V_1,V_2,V_3,V_4)\}_{n\geq 0}$ is defined by the fifth order recurrence relations

$$V_n=V_{n-1}+V_{n-2}+V_{n-3}+V_{n-4}+2V_{n-5} \tag{9}$$

with the initial values $V_0=c_0, V_1=c_1, V_2=c_2, V_3=c_3, V_4=c_4$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n}=-\frac{1}{2}V_{-(n-1)}-\frac{1}{2}V_{-(n-2)}-\frac{1}{2}V_{-(n-3)}-\frac{1}{2}V_{-(n-4)}+\frac{1}{2}V_{-(n-5)}$$

for $n=1,2,3,\ldots$. Therefore, recurrence (9) holds for all integer n. For more information on the generalized fifth order Jacobsthal numbers, see [7].

The first few generalized fifth order Jacobsthal numbers with positive subscript and negative subscript are given in the Table 1

Eq. (3) can be used to obtain Binet's formula of generalized fifth order Jacobsthal numbers. Generalized fifth order Jacobsthal numbers can be expressed, for all integers n, using Binet's formula

$$\begin{aligned} V_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)}+\frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} \\ &\quad +\frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}+\frac{p_5\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}, \end{aligned}$$

where

$$\begin{aligned}
p_1&=V_4-(\beta+\gamma+\delta+\lambda)V_3+(\beta\lambda+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta)V_2-(\beta\lambda\gamma+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta)V_1+(\beta\lambda\gamma\delta)V_0,\\
p_2&=V_4-(\alpha+\gamma+\delta+\lambda)V_3+(\alpha\lambda+\alpha\gamma+\alpha\delta+\lambda\gamma+\lambda\delta+\gamma\delta)V_2-(\alpha\lambda\gamma+\alpha\lambda\delta+\alpha\gamma\delta+\lambda\gamma\delta)V_1+(\alpha\lambda\gamma\delta)V_0,\\
p_3&=V_4-(\alpha+\beta+\delta+\lambda)V_3+(\alpha\beta+\alpha\lambda+\beta\lambda+\alpha\delta+\beta\delta+\lambda\delta)V_2-(\alpha\beta\lambda+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\delta)V_1+(\alpha\beta\lambda\delta)V_0,\\
p_4&=V_4-(\alpha+\beta+\gamma+\lambda)V_3+(\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\beta\gamma+\lambda\gamma)V_2-(\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\beta\lambda\gamma)V_1+(\alpha\beta\lambda\gamma)V_0,\\
p_5&=V_4-(\alpha+\beta+\gamma+\delta)V_3+(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)V_2-(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)V_1+(\alpha\beta\gamma\delta)V_0.
\end{aligned}$$

As $\{V_n\}$ is a fifth order recurrence sequence (difference equation), it's characteristic equation is

$$x^5-x^4-x^3-x^2-x-2=(x-2)(x^4+x^3+x^2+x+1)=0. \tag{10}$$

The roots $\alpha,\beta,\gamma,\delta$ and λ of Eq. (10) are given by:

$$\alpha = 2,$$

$$\begin{aligned}
\beta &= \frac{1}{4}(\sqrt{5}-1)+\frac{\sqrt{2\sqrt{5}+10}}{4}i,\\
\gamma &= \frac{1}{4}(\sqrt{5}-1)-\frac{\sqrt{2\sqrt{5}+10}}{4}i,\\
\delta &= -\frac{1}{4}(\sqrt{5}+1)+\frac{\sqrt{-2\sqrt{5}+10}}{4}i,\\
\lambda &= -\frac{1}{4}(\sqrt{5}+1)-\frac{\sqrt{-2\sqrt{5}+10}}{4}i.
\end{aligned}$$

Note that we have the following identities:

$$\begin{aligned}
\alpha+\beta+\gamma+\delta+\lambda &= 1,\\
\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\alpha\delta+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta &= -1,\\
\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\gamma+\alpha\gamma\delta+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta &= 1,\\
\alpha\beta\lambda\gamma+\alpha\beta\lambda\delta+\alpha\beta\gamma\delta+\alpha\lambda\gamma\delta+\beta\lambda\gamma\delta &= -1,\\
\alpha\beta\gamma\delta\lambda &= 2.
\end{aligned}$$

Now we consider four special cases of the sequence $\{V_n\}$. Fifth-order Jacobsthal sequence $\{J_n\}_{n\geq 0}$, fifth order Jacobsthal-Lucas sequence $\{j_n\}_{n\geq 0}$, adjusted fifth order Jacobsthal sequence $\{S_n\}_{n\geq 0}$ and modified fifth order Jacobsthal-Lucas sequence $\{R_n\}_{n\geq 0}$ are defined, respectively, by the fifth order recurrence relations

$$J_{n+5} = J_{n+4}+J_{n+3}+J_{n+2}+J_{n+1}+2J_n, J_0=0, J_1=1, J_2=1, J_3=1, J_4=1, \tag{11}$$

$$j_{n+5} = j_{n+4}+j_{n+3}+j_{n+2}+j_{n+1}+2j_n,\ j_0=2, j_1=1, j_2=5, j_3=10, j_4=20, \tag{12}$$

$$S_{n+5} = S_{n+4}+S_{n+3}+S_{n+2}+S_{n+1}+2S_n, S_0=0, S_1=1, S_2=1, S_3=2, S_4=4, \tag{13}$$

$$R_{n+5} = R_{n+4}+R_{n+3}+R_{n+2}+R_{n+1}+2R_n, R_0=5, R_1=1, R_2=3, R_3=7, R_4=19. \tag{14}$$

The sequences $\{J_n\}_{n\geq 0}$, $\{j_n\}_{n\geq 0}$, $\{S_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}
J_{-n} &= -\frac{1}{2}J_{-(n-1)}-\frac{1}{2}J_{-(n-2)}-\frac{1}{2}J_{-(n-3)}-J_{-(n-4)}+\frac{1}{2}J_{-(n-5)},\\
j_{-n} &= -\frac{1}{2}j_{-(n-1)}-\frac{1}{2}j_{-(n-2)}-\frac{1}{2}j_{-(n-3)}-\frac{1}{2}j_{-(n-4)}+\frac{1}{2}j_{-(n-5)},\\
S_{-n} &= -\frac{1}{2}S_{-(n-1)}-\frac{1}{2}S_{-(n-2)}-\frac{1}{2}S_{-(n-3)}-\frac{1}{2}S_{-(n-4)}+\frac{1}{2}S_{-(n-5)},\\
R_{-n} &= -\frac{1}{2}R_{-(n-1)}-\frac{1}{2}R_{-(n-2)}-\frac{1}{2}R_{-(n-3)}-\frac{1}{2}R_{-(n-4)}+\frac{1}{2}R_{-(n-5)},
\end{aligned}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences (11)-(14) hold for all integer n.

Next, we present the first few values of the fifth order Jacobsthal, fifth order Jacobsthal-Lucas, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas numbers with positive and negative subscripts in the following Table 2:

Table 2. The first few values of the special fifth order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
J_n	0	1	1	1	1	4	9	17	33	65	132	265	529	1057
J_{-n}		-1	0	$\frac{1}{2}$	$\frac{3}{4}$	$-\frac{1}{8}$	$-\frac{17}{16}$	$-\frac{1}{32}$	$\frac{31}{64}$	$\frac{95}{128}$	$-\frac{33}{256}$	$-\frac{545}{512}$	$-\frac{33}{1024}$	$\frac{991}{2048}$
j_n	2	1	5	10	20	40	77	157	314	628	1256	2509	5021	10042
j_{-n}		1	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{11}{8}$	$\frac{13}{16}$	$\frac{13}{32}$	$\frac{13}{64}$	$\frac{13}{128}$	$-\frac{371}{256}$	$\frac{397}{512}$	$\frac{397}{1024}$	$\frac{397}{2048}$	$\frac{397}{4096}$
S_n	0	1	1	2	4	8	17	33	66	132	264	529	1057	2114
S_{-n}		0	0	0	$\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$-\frac{1}{16}$	$-\frac{1}{32}$	$\frac{31}{64}$	$-\frac{33}{128}$	$-\frac{33}{256}$	$-\frac{33}{512}$	$-\frac{33}{1024}$
R_n	5	1	3	7	15	36	63	127	255	511	1028	2047	4095	8191
R_{-n}		$-\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{7}{8}$	$-\frac{15}{16}$	$\frac{129}{32}$	$-\frac{63}{64}$	$-\frac{127}{128}$	$-\frac{255}{256}$	$-\frac{511}{512}$	$\frac{4097}{1024}$	$-\frac{2047}{2048}$	$-\frac{4095}{4096}$	$-\frac{8191}{8192}$

For all integers n, Binet formulas of fifth order Jacobsthal, fifth order Jacobsthal-Lucas, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas numbers are

$$\begin{aligned} J_n &= \frac{(\alpha^3-\alpha-2)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{(\beta^3-\beta-2)\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ &\quad + \frac{(\gamma^3-\gamma-2)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} + \frac{(\delta^3-\delta-2)\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} \\ &\quad + \frac{(\lambda^3-\lambda-2)\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}, \end{aligned}$$

$$\begin{aligned} j_n &= \frac{(\alpha^4+4\alpha^3+4\alpha^2+4\alpha+4)\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{(\beta^4+4\beta^3+4\beta^2+4\beta+4)\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ &\quad + \frac{(\gamma^4+4\gamma^3+4\gamma^2+4\gamma+4)\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} + \frac{(\delta^4+4\delta^3+4\delta^2+4\delta+4)\delta^{n-1}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} \\ &\quad + \frac{(\lambda^4+4\lambda^3+4\lambda^2+4\lambda+4)\lambda^{n-1}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}, \end{aligned}$$

$$\begin{aligned} S_n &= \frac{\alpha^{n+3}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{\beta^{n+3}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ &\quad + \frac{\gamma^{n+3}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} + \frac{\delta^{n+3}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} \\ &\quad + \frac{\lambda^{n+3}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}, \\ R_n &= \alpha^n+\beta^n+\gamma^n+\delta^n+\lambda^n \end{aligned}$$

respectively.

Binet formulas of fifth order Jacobsthal, fifth order Jacobsthal-Lucas, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas numbers can be given in the following forms:

$$\begin{aligned} J_n &= \frac{4}{31}\alpha^n - \frac{1}{155}((6\sqrt{5}+5)+2\sqrt{2}\sqrt{\sqrt{5}+5}(6+\sqrt{5})i)\beta^n \\ &\quad + \frac{1}{155}(-(6\sqrt{5}+5)+2\sqrt{2}\sqrt{\sqrt{5}+5}(6+\sqrt{5})i)\gamma^n \\ &\quad + \frac{1}{155}((6\sqrt{5}-5)+2\sqrt{2}\sqrt{5-\sqrt{5}}(\sqrt{5}-6)i)\delta^n \\ &\quad + \frac{1}{155}((6\sqrt{5}-5)+2\sqrt{2}\sqrt{5-\sqrt{5}}(-\sqrt{5}+6)i)\lambda^n, \end{aligned}$$

$$\begin{aligned} j_n &= \frac{38}{31}\alpha^n + \frac{1}{1240}(12(20-7\sqrt{5})+\sqrt{\sqrt{5}+5}(111\sqrt{2}+3\sqrt{10})i)\beta^n \\ &\quad + \frac{1}{1240}(12(20-7\sqrt{5})-\sqrt{\sqrt{5}+5}(111\sqrt{2}+3\sqrt{10})i)\gamma^n \\ &\quad + \frac{1}{1240}(12(20+7\sqrt{5})+\sqrt{5-\sqrt{5}}(111\sqrt{2}-3\sqrt{10})i)\delta^n \\ &\quad + \frac{1}{1240}(12(20+7\sqrt{5})+\sqrt{5-\sqrt{5}}(-111\sqrt{2}+3\sqrt{10})i)\lambda^n, \end{aligned}$$

$$\begin{aligned} S_n &= \frac{8}{31}\alpha^n + \frac{1}{1240}(4(-20+7\sqrt{5})-\sqrt{2}\sqrt{\sqrt{5}+5}(37i+i\sqrt{5}))\beta^n \\ &\quad + \frac{1}{1240}(4(-20+7\sqrt{5})+\sqrt{2}\sqrt{\sqrt{5}+5}(37i+i\sqrt{5}))\gamma^n \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{1240}(-4(20+7\sqrt{5})+\sqrt{2}\sqrt{\sqrt{5}+5}(21-19\sqrt{5})i)\delta^{n} \\
&+\frac{1}{1240}(-4(20+7\sqrt{5})+\sqrt{2}\sqrt{\sqrt{5}+5}(-21+19\sqrt{5})i)\lambda^{n}, \\
R_n \;&=\; \alpha^{n}+\beta^{n}+\gamma^{n}+\delta^{n}+\lambda^{n}.
\end{aligned}
$$

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} V_n x^n$ of the sequence V_n.

Lemma 2. *Suppose that $f_{V_n}(x)=\sum\limits_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized fifth order Jacobsthal sequence $\{V_n\}_{n\geq 0}$. Then, $\sum\limits_{n=0}^{\infty} V_n x^n$ is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0+(V_1-V_0)x+(V_2-V_1-V_0)x^2+(V_3-V_2-V_1-V_0)x^3+(V_4-V_3-V_2-V_1-V_0)x^4}{1-x-x^2-x^3-x^4-2x^5}.$$

The previous Lemma gives the following results as particular examples: generating function of the fifth order Jacobsthal, fifth order Jacobsthal-Lucas, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas are

$$
\begin{aligned}
f_{J_n}(x) &= \sum_{n=0}^{\infty} J_n x^n = \frac{x-x^3-2x^4}{1-x-x^2-x^3-x^4-2x^5}, \\
f_{j_n}(x) &= \sum_{n=0}^{\infty} j_n x^n = \frac{2-x+2x^2+2x^3+2x^4}{1-x-x^2-x^3-x^4-2x^5}, \\
f_{S_n}(x) &= \sum_{n=0}^{\infty} S_n x^n = \frac{x}{1-x-x^2-x^3-x^4-2x^5}, \\
f_{R_n}(x) &= \sum_{n=0}^{\infty} R_n x^n = \frac{5-4x-3x^2-2x^3-x^4}{1-x-x^2-x^3-x^4-2x^5},
\end{aligned}
$$

respectively.

2. Binomial transform of the generalized fifth order Jacobsthal sequence V_n

In [8], p. 137, Knuth introduced the idea of the binomial transform. Given a sequence of numbers (a_n), its binomial transform $(\hat{a}_n)$ may be defined by the rule

$$\hat{a}_n=\sum_{i=0}^{n}\binom{n}{i}a_i, \text{ with inversion } a_n=\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\hat{a}_i,$$

or, in the symmetric version

$$\hat{a}_n=\sum_{i=0}^{n}\binom{n}{i}(-1)^{i+1}a_i, \text{ with inversion } a_n=\sum_{i=0}^{n}\binom{n}{i}(-1)^{i+1}\hat{a}_i.$$

For more information on binomial transform, see, for example, [9–12] and references therein. For recent works on binomial transform of well-known sequences, see for example, [13–25].

In this section, we define the binomial transform of the generalized fifth order Jacobsthal sequence V_n and as special cases the binomial transform of the fifth order Jacobsthal, fifth order Jacobsthal-Lucas, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas sequences will be introduced.

Definition 1. The binomial transform of the generalized fifth order Jacobsthal sequence V_n is defined by

$$b_n=\widehat{V}_n=\sum_{i=0}^{n}\binom{n}{i}V_i.$$

The few terms of b_n are

$$\begin{aligned} b_0 &= \sum_{i=0}^{0}\binom{0}{i}V_i = V_0, \\ b_1 &= \sum_{i=0}^{1}\binom{1}{i}V_i = V_0 + V_1, \\ b_2 &= \sum_{i=0}^{2}\binom{2}{i}V_i = V_0 + 2V_1 + V_2, \\ b_3 &= \sum_{i=0}^{3}\binom{3}{i}V_i = V_0 + 3V_1 + 3V_2 + V_3, \\ b_4 &= \sum_{i=0}^{4}\binom{4}{i}V_i = V_0 + 4V_1 + 6V_2 + 4V_3 + V_4. \end{aligned}$$

Translated to matrix language, b_n has the nice (lower-triangular matrix) form

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ \vdots \end{pmatrix}.$$

As special cases of $b_n = \widehat{V}_n$, the binomial transforms of the fifth order Jacobsthal and fifth order Jacobsthal-Lucas sequences are defined as follows: The binomial transform of the fifth order Jacobsthal sequence J_n is

$$\widehat{J}_n = \sum_{i=0}^{n}\binom{n}{i}J_i,$$

and the binomial transform of the fifth order Jacobsthal-Lucas sequence j_n is

$$\widehat{j}_n = \sum_{i=0}^{n}\binom{n}{i}j_i,$$

The binomial transform of the adjusted fifth order Jacobsthal sequence S_n is

$$\widehat{S}_n = \sum_{i=0}^{n}\binom{n}{i}S_i,$$

and the binomial transform of the modified fifth order Jacobsthal-Lucas sequence R_n is

$$\widehat{R}_n = \sum_{i=0}^{n}\binom{n}{i}R_i.$$

Lemma 3. *For $n \geq 0$, the binomial transform of the generalized fifth order Jacobsthal sequence V_n satisfies the following relation:*

$$b_{n+1} = \sum_{i=0}^{n}\binom{n}{i}(V_i + V_{i+1}).$$

Proof. We use the following well-known identity:

$$\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}.$$

Note also that

$$\binom{n+1}{0} = \binom{n}{0} = 1 \text{ and } \binom{n}{n+1} = 0.$$

Then

$$\begin{aligned} b_{n+1} &= V_0 + \sum_{i=1}^{n+1} \binom{n+1}{i} V_i \\ &= V_0 + \sum_{i=1}^{n+1} \binom{n}{i} V_i + \sum_{i=1}^{n+1} \binom{n}{i-1} V_i \\ &= V_0 + \sum_{i=1}^{n} \binom{n}{i} V_i + \sum_{i=0}^{n} \binom{n}{i} V_{i+1} \\ &= \sum_{i=0}^{n} \binom{n}{i} V_i + \sum_{i=0}^{n} \binom{n}{i} V_{i+1} \\ &= \sum_{i=0}^{n} \binom{n}{i} (V_i + V_{i+1}). \end{aligned}$$

This completes the proof. □

Remark 1. From the Lemma 3, we see that

$$b_{n+1} = b_n + \sum_{i=0}^{n} \binom{n}{i} V_{i+1}.$$

The following theorem gives recurrent relations of the binomial transform of the generalized fifth order Jacobsthal sequence.

Theorem 4. *For $n \geq 0$, the binomial transform of the generalized fifth order Jacobsthal sequence V_n satisfies the following recurrence relation:*

$$b_{n+5} = 6b_{n+4} - 13b_{n+3} + 14b_{n+2} - 7b_{n+1} + 3b_n \,. \tag{15}$$

Proof. To show (15), writing

$$b_{n+5} = r_1 \times b_{n+4} + s_1 \times b_{n+3} + t_1 \times b_{n+2} + u_1 \times b_{n+1} + v_1 \times b_n$$

and taking the values $n = 0, 1, 2, 3, 4$ and then solving the system of equations

$$\begin{aligned} b_5 &= r_1 \times b_4 + s_1 \times b_3 + t_1 \times b_2 + u_1 \times b_1 + v_1 \times b_0 \\ b_6 &= r_1 \times b_5 + s_1 \times b_4 + t_1 \times b_3 + u_1 \times b_2 + v_1 \times b_1 \\ b_7 &= r_1 \times b_6 + s_1 \times b_5 + t_1 \times b_4 + u_1 \times b_3 + v_1 \times b_2 \\ b_8 &= r_1 \times b_7 + s_1 \times b_6 + t_1 \times b_5 + u_1 \times b_4 + v_1 \times b_3 \\ b_9 &= r_1 \times b_8 + s_1 \times b_7 + t_1 \times b_6 + u_1 \times b_5 + v_1 \times b_4 \end{aligned}$$

we find that $r_1 = 6, s_1 = -13, t_1 = 14, u_1 = -7, v_1 = 3$.

□

The sequence $\{b_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$b_{-n} = \frac{7}{3} b_{-(n-1)} - \frac{14}{3} b_{-(n-2)} + \frac{13}{3} b_{-(n-3)} - 2b_{-(n-4)} + \frac{1}{3} b_{-(n-5)}$$

for $n = 1, 2, 3, ...$. Therefore, recurrence (15) holds for all integer n.

Note that the recurence relation (15) is independent from initial values. So,

$$\begin{aligned} \widehat{J}_{n+5} &= 6\widehat{J}_{n+4} - 13\widehat{J}_{n+3} + 14\widehat{J}_{n+2} - 7\widehat{J}_{n+1} + 3\widehat{J}_n, \\ \widehat{j}_{n+5} &= 6\widehat{j}_{n+4} - 13\widehat{j}_{n+3} + 14\widehat{j}_{n+2} - 7\widehat{j}_{n+1} + 3\widehat{j}_n, \\ \widehat{S}_{n+5} &= 6\widehat{S}_{n+4} - 13\widehat{S}_{n+3} + 14\widehat{S}_{n+2} - 7\widehat{S}_{n+1} + 3\widehat{S}_n, \end{aligned}$$

$$\widehat{R}_{n+5} = 6\widehat{R}_{n+4} - 13\widehat{R}_{n+3} + 14\widehat{R}_{n+2} - 7\widehat{R}_{n+1} + 3\widehat{R}_{n}.$$

The first few terms of the binomial transform of the generalized fifth order Jacobsthal sequence with positive subscript and negative subscript are given in the following Table 3.

Table 3. A few binomial transform (terms) of the generalized fifth order Jacobsthal sequence.

n	b_n	b_{-n}
0	V_0	V_0
1	$V_0 + V_1$	$\frac{1}{3}(V_0 - 2V_1 + V_2 - 2V_3 + V_4)$
2	$V_0 + 2V_1 + V_2$	$-\frac{1}{9}(11V_0 + 2V_1 + 2V_2 + 11V_3 - 7V_4)$
3	$V_0 + 3V_1 + 3V_2 + V_3$	$-\frac{1}{27}(47V_0 - 34V_1 + 47V_2 - 7V_3 - 7V_4)$
4	$V_0 + 4V_1 + 6V_2 + 4V_3 + V_4$	$\frac{1}{81}(115V_0 + 115V_1 - 128V_2 + 277V_3 - 128V_4)$
5	$3V_0 + 6V_1 + 11V_2 + 11V_3 + 6V_4$	$\frac{1}{243}(1411V_0 - 533V_1 + 682V_2 + 682V_3 - 533V_4)$
6	$15V_0 + 15V_1 + 23V_2 + 28V_3 + 23V_4$	$\frac{1}{729}(1411V_0 - 4421V_1 + 5056V_2 - 4421V_3 + 1411V_4)$
7	$61V_0 + 53V_1 + 61V_2 + 74V_3 + 74V_4$	$-\frac{1}{2187}(29207V_0 + 776V_1 + 776V_2 + 29207V_3 - 16720V_4)$

The first few terms of the binomial transform numbers of the fifth order Jacobsthal, fifth order Jacobsthal-Lucas, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas sequences with positive subscript and negative subscript are given in the following Table 4.

Table 4. A few binomial transform (terms).

n	0	1	2	3	4	5	6	7	8	9	10	11
$\widehat{J}_n$	0	1	3	7	15	34	89	262	807	2489	7590	22914
$\widehat{J}_{-n}$		$-\frac{2}{3}$	$-\frac{8}{9}$	$\frac{1}{27}$	$\frac{136}{81}$	$\frac{298}{243}$	$-\frac{2375}{729}$	$-\frac{14039}{2187}$	$\frac{14392}{6561}$	$\frac{375247}{19683}$	$\frac{788590}{59049}$	$-\frac{6474437}{177147}$
$\widehat{j}_n$	2	3	9	30	96	297	900	2700	8076	24165	72393	217077
$\widehat{j}_{-n}$		$\frac{5}{3}$	$-\frac{4}{9}$	$-\frac{85}{27}$	$-\frac{85}{81}$	$\frac{1859}{243}$	$\frac{7691}{729}$	$-\frac{20740}{2187}$	$-\frac{243814}{6561}$	$-\frac{243814}{19683}$	$\frac{5011547}{59049}$	$\frac{20777630}{177147}$
$\widehat{S}_n$	0	1	3	8	22	63	186	558	1682	5067	15235	45739
$\widehat{S}_{-n}$		$-\frac{1}{3}$	$\frac{2}{9}$	$\frac{29}{27}$	$\frac{29}{81}$	$-\frac{619}{243}$	$-\frac{2563}{729}$	$\frac{6914}{2187}$	$\frac{81272}{6561}$	$\frac{81272}{19683}$	$-\frac{1670515}{59049}$	$-\frac{6925876}{177147}$
$\widehat{R}_n$	5	6	10	24	70	221	700	2169	6590	19806	59295	177469
$\widehat{R}_{-n}$		$\frac{7}{3}$	$-\frac{35}{9}$	$-\frac{188}{27}$	$\frac{325}{81}$	$\frac{5347}{243}$	$\frac{8020}{729}$	$-\frac{102788}{2187}$	$-\frac{498635}{6561}$	$\frac{925102}{19683}$	$\frac{14526055}{59049}$	$\frac{21789082}{177147}$

Eq. (3) can be used to obtain Binet's formula of the binomial transform of generalized fifth order Jacobsthal numbers. Binet's formula of the binomial transform of generalized fifth order Jacobsthal numbers can be given as

$$\begin{aligned} b_n &= \frac{C_1\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_1-\theta_4)(\theta_1-\theta_5)} + \frac{C_2\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)(\theta_2-\theta_4)(\theta_2-\theta_5)} \\ &\quad + \frac{C_3\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)(\theta_3-\theta_4)(\theta_3-\theta_5)} + \frac{C_4\theta_4^n}{(\theta_4-\theta_1)(\theta_4-\theta_2)(\theta_4-\theta_3)(\theta_4-\theta_5)} \\ &\quad + \frac{C_5\theta_5^n}{(\theta_5-\theta_1)(\theta_5-\theta_2)(\theta_5-\theta_3)(\theta_5-\theta_4)}, \end{aligned} \tag{16}$$

where

$$\begin{aligned} C_1 &= b_4 - (\theta_2+\theta_3+\theta_4+\theta_5)b_3 + (\theta_2\theta_5+\theta_2\theta_3+\theta_5\theta_3+\theta_2\theta_4+\theta_5\theta_4+\theta_3\theta_4)b_2 \\ &\quad -(\theta_2\theta_5\theta_3+\theta_2\theta_5\theta_4+\theta_2\theta_3\theta_4+\theta_5\theta_3\theta_4)b_1 + (\theta_2\theta_5\theta_3\theta_4)b_0, \\ C_2 &= b_4 - (\theta_1+\theta_3+\theta_4+\theta_5)b_3 + (\theta_1\theta_5+\theta_1\theta_3+\theta_1\theta_4+\theta_5\theta_3+\theta_5\theta_4+\theta_3\theta_4)b_2 \\ &\quad -(\theta_1\theta_5\theta_3+\theta_1\theta_5\theta_4+\theta_1\theta_3\theta_4+\theta_5\theta_3\theta_4)b_1 + (\theta_1\theta_5\theta_3\theta_4)b_0, \end{aligned}$$

$$
\begin{aligned}
C_3 &= b_4-(\theta_1+\theta_2+\theta_4+\theta_5)b_3+(\theta_1\theta_2+\theta_1\theta_5+\theta_2\theta_5+\theta_1\theta_4+\theta_2\theta_4+\theta_5\theta_4)b_2 \\
&\quad -(\theta_1\theta_2\theta_5+\theta_1\theta_2\theta_4+\theta_1\theta_5\theta_4+\theta_2\theta_5\theta_4)b_1+(\theta_1\theta_2\theta_5\theta_4)b_0, \\
C_4 &= b_4-(\theta_1+\theta_2+\theta_3+\theta_5)b_3+(\theta_1\theta_2+\theta_1\theta_5+\theta_1\theta_3+\theta_2\theta_5+\theta_2\theta_3+\theta_5\theta_3)b_2 \\
&\quad -(\theta_1\theta_2\theta_5+\theta_1\theta_2\theta_3+\theta_1\theta_5\theta_3+\theta_2\theta_5\theta_3)b_1+(\theta_1\theta_2\theta_5\theta_3)b_0, \\
C_5 &= b_4-(\theta_1+\theta_2+\theta_3+\theta_4)b_3+(\theta_1\theta_2+\theta_1\theta_3+\theta_1\theta_4+\theta_2\theta_3+\theta_2\theta_4+\theta_3\theta_4)b_2 \\
&\quad -(\theta_1\theta_2\theta_3+\theta_1\theta_2\theta_4+\theta_1\theta_3\theta_4+\theta_2\theta_3\theta_4)b_1+(\theta_1\theta_2\theta_3\theta_4)b_0.
\end{aligned}
$$

Here, $\theta_1,\theta_2,\theta_3,\theta_4$ and θ_5 are the roots of the equation

$$x^5-6x^4+13x^3-14x^2+7x-3=(x-3)\left(x^4-3x^3+4x^2-2x+1\right)=0.$$

Moreover, the approximate value of $\theta_1,\theta_2,\theta_3,\theta_4$ and θ_5 are given by

$$
\begin{aligned}
\theta_1 &= 3, \\
\theta_2 &= 1.30901699437495+0.951056516295154i, \\
\theta_3 &= 1.30901699437495-0.951056516295154i, \\
\theta_4 &= 0.190983005625053+0.587785252292473i, \\
\theta_5 &= 0.190983005625053-0.587785252292473i.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\theta_1+\theta_2+\theta_3+\theta_4+\theta_5 &= 6, \\
\theta_1\theta_2+\theta_1\theta_5+\theta_1\theta_3+\theta_2\theta_5+\theta_1\theta_4+\theta_2\theta_3+\theta_5\theta_3+\theta_2\theta_4+\theta_5\theta_4+\theta_3\theta_4 &= 13, \\
\theta_1\theta_2\theta_5+\theta_1\theta_2\theta_3+\theta_1\theta_5\theta_3+\theta_1\theta_2\theta_4+\theta_1\theta_5\theta_4+\theta_2\theta_5\theta_3+\theta_1\theta_3\theta_4+\theta_2\theta_5\theta_4+\theta_2\theta_3\theta_4+\theta_5\theta_3\theta_4 &= 14, \\
\theta_1\theta_2\theta_5\theta_3+\theta_1\theta_2\theta_5\theta_4+\theta_1\theta_2\theta_3\theta_4+\theta_1\theta_5\theta_3\theta_4+\theta_2\theta_5\theta_3\theta_4 &= 7, \\
\theta_1\theta_2\theta_3\theta_4\theta_5 &= 3.
\end{aligned}
$$

For all integers n, (Binet's formulas of) binomial transforms of fifth order Jacobsthal, fifth order Jacobsthal-Lucas, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas numbers (using initial conditions in (16)) can be expressed using Binet's formulas as

$$
\begin{aligned}
\widehat{J}_n &= \frac{(\theta_1^3-3\theta_1^2+2\theta_1-2)\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_1-\theta_4)(\theta_1-\theta_5)}+\frac{(\theta_2^3-3\theta_2^2+2\theta_2-2)\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)(\theta_2-\theta_4)(\theta_2-\theta_5)} \\
&\quad +\frac{(\theta_3^3-3\theta_3^2+2\theta_3-2)\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)(\theta_3-\theta_4)(\theta_3-\theta_5)}+\frac{(\theta_4^3-3\theta_4^2+2\theta_4-2)\theta_4^n}{(\theta_4-\theta_1)(\theta_4-\theta_2)(\theta_4-\theta_3)(\theta_4-\theta_5)} \\
&\quad +\frac{(\theta_5^3-3\theta_5^2+2\theta_5-2)\theta_5^n}{(\theta_5-\theta_1)(\theta_5-\theta_2)(\theta_5-\theta_3)(\theta_5-\theta_4)},
\end{aligned}
$$

$$
\begin{aligned}
\widehat{j}_n &= \frac{(2\theta_1^4-9\theta_1^3+17\theta_1^2-13\theta_1+5)\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_1-\theta_4)(\theta_1-\theta_5)}+\frac{(2\theta_2^4-9\theta_2^3+17\theta_2^2-13\theta_2+5)\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)(\theta_2-\theta_4)(\theta_2-\theta_5)} \\
&\quad +\frac{(2\theta_3^4-9\theta_3^3+17\theta_3^2-13\theta_3+5)\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)(\theta_3-\theta_4)(\theta_3-\theta_5)}+\frac{(2\theta_4^4-9\theta_4^3+17\theta_4^2-13\theta_4+5)\theta_4^n}{(\theta_4-\theta_1)(\theta_4-\theta_2)(\theta_4-\theta_3)(\theta_4-\theta_5)} \\
&\quad +\frac{(2\theta_5^4-9\theta_5^3+17\theta_5^2-13\theta_5+5)\theta_5^n}{(\theta_5-\theta_1)(\theta_5-\theta_2)(\theta_5-\theta_3)(\theta_5-\theta_4)},
\end{aligned}
$$

$$
\begin{aligned}
\widehat{S}_n &= \frac{(\theta_1-1)^3\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_1-\theta_4)(\theta_1-\theta_5)}+\frac{(\theta_2-1)^3\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)(\theta_2-\theta_4)(\theta_2-\theta_5)} \\
&\quad +\frac{(\theta_3-1)^3\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)(\theta_3-\theta_4)(\theta_3-\theta_5)}+\frac{(\theta_4-1)^3\theta_4^n}{(\theta_4-\theta_1)(\theta_4-\theta_2)(\theta_4-\theta_3)(\theta_4-\theta_5)}
\end{aligned}
$$

$$
\begin{aligned}
& + \frac{(\theta_5-1)^3\theta_5^n}{(\theta_5-\theta_1)(\theta_5-\theta_2)(\theta_5-\theta_3)(\theta_5-\theta_4)}, \\
\widehat{R}_n &= \theta_1^n+\theta_2^n+\theta_3^n+\theta_4^n+\theta_5^n,
\end{aligned}
$$

respectively.

3. Generating functions and obtaining Binet formula of binomial transform from generating function

The generating function of the binomial transform of the generalized fifth order Jacobsthal sequence V_n is a power series centered at the origin whose coefficients are the binomial transform of the generalized fifth order Jacobsthal sequence.

Next, we give the ordinary generating function $f_{b_n}(x)=\sum\limits_{n=0}^{\infty} b_n x^n$ of the sequence b_n.

Lemma 4. *Suppose that $f_{b_n}(x)=\sum\limits_{n=0}^{\infty} b_n x^n$ is the ordinary generating function of the binomial transform of the generalized fifth order Jacobsthal sequence $\{V_n\}_{n\geq 0}$. Then, $f_{b_n}(x)$ is given by*

$$
f_{b_n}(x)=\frac{V_0+(V_1-5V_0)x+(8V_0-4V_1+V_2)x^2+(4V_1-6V_0-3V_2+V_3)x^3+(V_0-2V_1+V_2-2V_3+V_4)x^4}{1-6x+13x^2-14x^3+7x^4-3x^5}. \tag{17}
$$

Proof. Using Lemma 1, we obtain

$$
\begin{aligned}
f_{b_n}(x) &= \frac{b_0+(b_1-6b_0)x+(b_2-6b_1+13b_0)x^2+(b_3-6b_2+13b_1-14b_0)x^3+(b_4-6b_3+13b_2-14b_1+7b_0)x^4}{1-6x+13x^2-14x^3+7x^4-3x^5} \\
&= \frac{V_0+(V_1-5V_0)x+(8V_0-4V_1+V_2)x^2+(4V_1-6V_0-3V_2+V_3)x^3+(V_0-2V_1+V_2-2V_3+V_4)x^4}{1-6x+13x^2-14x^3+7x^4-3x^5},
\end{aligned}
$$

where

$$
\begin{aligned}
b_0 &= V_0, \\
b_1 &= V_0+V_1, \\
b_2 &= V_0+2V_1+V_2, \\
b_3 &= V_0+3V_1+3V_2+V_3, \\
b_4 &= V_0+4V_1+6V_2+4V_3+V_4.
\end{aligned}
$$

□

Note that P. Barry shows in [26] that if $A(x)$ is the generating function of the sequence $\{a_n\}$, then

$$
S(x)=\frac{1}{1-x}A\left(\frac{x}{1-x}\right)
$$

is the generating function of the sequence $\{b_n\}$ with $b_n=\sum\limits_{i=0}^{n}\binom{n}{i}a_i$. In our case, since

$$
A(x)=\frac{V_0+(V_1-V_0)x+(V_2-V_1-V_0)x^2+(V_3-V_2-V_1-V_0)x^3+(V_4-V_3-V_2-V_1-V_0)x^4}{1-x-x^2-x^3-x^4-2x^5}, \quad \text{see Lemma 2,}
$$

we obtain

$$
\begin{aligned}
S(x) &= \frac{1}{1-x}A\left(\frac{x}{1-x}\right) \\
&= \frac{V_0+(V_1-5V_0)x+(8V_0-4V_1+V_2)x^2+(4V_1-6V_0-3V_2+V_3)x^3+(V_0-2V_1+V_2-2V_3+V_4)x^4}{1-6x+13x^2-14x^3+7x^4-3x^5}.
\end{aligned}
$$

The Lemma 4 gives the following results as particular examples.

Corollary 3. *Generating functions of the binomial transform of the fifth order Jacobsthal, fifth order Jacobsthal-Lucas, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas numbers are*

$$\begin{aligned}
\sum_{n=0}^{\infty} \widehat{J}_n x^n &= \frac{x-3x^2+2x^3-2x^4}{1-6x+13x^2-14x^3+7x^4-3x^5}, \\
\sum_{n=0}^{\infty} \widehat{j}_n x^n &= \frac{2-9x+17x^2-13x^3+5x^4}{1-6x+13x^2-14x^3+7x^4-3x^5}, \\
\sum_{n=0}^{\infty} \widehat{S}_n x^n &= \frac{x-3x^2+3x^3-x^4}{1-6x+13x^2-14x^3+7x^4-3x^5}, \\
\sum_{n=0}^{\infty} \widehat{R}_n x^n &= \frac{5-24x+39x^2-28x^3+7x^4}{1-6x+13x^2-14x^3+7x^4-3x^5}.
\end{aligned}$$

respectively.

4. Simson formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n,$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized Pentanacci sequence $\{W_n\}$.

Theorem 5 (Simson formula of generalized Pentanacci numbers)**.** *For all integers n, we have*

$$\begin{vmatrix} W_{n+4} & W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+3} & W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} & W_{n-3} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} & W_{n-4} \end{vmatrix} = v^n \begin{vmatrix} W_4 & W_3 & W_2 & W_1 & W_0 \\ W_3 & W_2 & W_1 & W_0 & W_{-1} \\ W_2 & W_1 & W_0 & W_{-1} & W_{-2} \\ W_1 & W_0 & W_{-1} & W_{-2} & W_{-3} \\ W_0 & W_{-1} & W_{-2} & W_{-3} & W_{-4} \end{vmatrix}. \tag{18}$$

Proof. Eq. (18) is given in [27], Theorem 3.1. □

Taking $\{W_n\} = \{b_n\}$ in the above theorem and considering $b_{n+5} = 6b_{n+4} - 13b_{n+3} + 14b_{n+2} - 7b_{n+1} + 3b_n$, $r = 6, s = -13, t = 14, u = -7, v = 3$, we have the following proposition.

Proposition 1. *For all integers n, Simson formula of binomial transforms of generalized fifth order Jacobsthal numbers is given as*

$$\begin{vmatrix} b_{n+4} & b_{n+3} & b_{n+2} & b_{n+1} & b_n \\ b_{n+3} & b_{n+2} & b_{n+1} & b_n & b_{n-1} \\ b_{n+2} & b_{n+1} & b_n & b_{n-1} & b_{n-2} \\ b_{n+1} & b_n & b_{n-1} & b_{n-2} & b_{n-3} \\ b_n & b_{n-1} & b_{n-2} & b_{n-3} & b_{n-4} \end{vmatrix} = 3^n \begin{vmatrix} b_4 & b_3 & b_2 & b_1 & b_0 \\ b_3 & b_2 & b_1 & b_0 & b_{-1} \\ b_2 & b_1 & b_0 & b_{-1} & b_{-2} \\ b_1 & b_0 & b_{-1} & b_{-2} & b_{-3} \\ b_0 & b_{-1} & b_{-2} & b_{-3} & b_{-4} \end{vmatrix}.$$

The Proposition 1 gives the following results as particular examples.

Corollary 4. *For all integers n, Simson formula of binomial transforms of the fifth order Jacobsthal, fifth order Jacobsthal-Lucas, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas numbers are given as*

$$\begin{vmatrix} \widehat{J}_{n+4} & \widehat{J}_{n+3} & \widehat{J}_{n+2} & \widehat{J}_{n+1} & \widehat{J}_{n} \\ \widehat{J}_{n+3} & \widehat{J}_{n+2} & \widehat{J}_{n+1} & \widehat{J}_{n} & \widehat{J}_{n-1} \\ \widehat{J}_{n+2} & \widehat{J}_{n+1} & \widehat{J}_{n} & \widehat{J}_{n-1} & \widehat{J}_{n-2} \\ \widehat{J}_{n+1} & \widehat{J}_{n} & \widehat{J}_{n-1} & \widehat{J}_{n-2} & \widehat{J}_{n-3} \\ \widehat{J}_{n} & \widehat{J}_{n-1} & \widehat{J}_{n-2} & \widehat{J}_{n-3} & \widehat{J}_{n-4} \end{vmatrix} = 44 \times 3^{n-4},$$

$$\begin{vmatrix} \widehat{j}_{n+4} & \widehat{j}_{n+3} & \widehat{j}_{n+2} & \widehat{j}_{n+1} & \widehat{j}_{n} \\ \widehat{j}_{n+3} & \widehat{j}_{n+2} & \widehat{j}_{n+1} & \widehat{j}_{n} & \widehat{j}_{n-1} \\ \widehat{j}_{n+2} & \widehat{j}_{n+1} & \widehat{j}_{n} & \widehat{j}_{n-1} & \widehat{j}_{n-2} \\ \widehat{j}_{n+1} & \widehat{j}_{n} & \widehat{j}_{n-1} & \widehat{j}_{n-2} & \widehat{j}_{n-3} \\ \widehat{j}_{n} & \widehat{j}_{n-1} & \widehat{j}_{n-2} & \widehat{j}_{n-3} & \widehat{j}_{n-4} \end{vmatrix} = 38 \times 3^{n},$$

$$\begin{vmatrix} \widehat{S}_{n+4} & \widehat{S}_{n+3} & \widehat{S}_{n+2} & \widehat{S}_{n+1} & \widehat{S}_{n} \\ \widehat{S}_{n+3} & \widehat{S}_{n+2} & \widehat{S}_{n+1} & \widehat{S}_{n} & \widehat{S}_{n-1} \\ \widehat{S}_{n+2} & \widehat{S}_{n+1} & \widehat{S}_{n} & \widehat{S}_{n-1} & \widehat{S}_{n-2} \\ \widehat{S}_{n+1} & \widehat{S}_{n} & \widehat{S}_{n-1} & \widehat{S}_{n-2} & \widehat{S}_{n-3} \\ \widehat{S}_{n} & \widehat{S}_{n-1} & \widehat{S}_{n-2} & \widehat{S}_{n-3} & \widehat{S}_{n-4} \end{vmatrix} = 8 \times 3^{n-4},$$

$$\begin{vmatrix} \widehat{R}_{n+4} & \widehat{R}_{n+3} & \widehat{R}_{n+2} & \widehat{R}_{n+1} & \widehat{R}_{n} \\ \widehat{R}_{n+3} & \widehat{R}_{n+2} & \widehat{R}_{n+1} & \widehat{R}_{n} & \widehat{R}_{n-1} \\ \widehat{R}_{n+2} & \widehat{R}_{n+1} & \widehat{R}_{n} & \widehat{R}_{n-1} & \widehat{R}_{n-2} \\ \widehat{R}_{n+1} & \widehat{R}_{n} & \widehat{R}_{n-1} & \widehat{R}_{n-2} & \widehat{R}_{n-3} \\ \widehat{R}_{n} & \widehat{R}_{n-1} & \widehat{R}_{n-2} & \widehat{R}_{n-3} & \widehat{R}_{n-4} \end{vmatrix} = 120125 \times 3^{n-4},$$

respectively.

5. Some identities

In this section, we obtain some identities of binomial transforms of fifth order Jacobsthal, fifth order Jacobsthal-Lucas, adjusted fifth order Jacobsthal and modified fifth order Jacobsthal-Lucas numbers. First, we present a few basic relations between $\{\widehat{J}_n\}$ and $\{\widehat{j}_n\}$.

Lemma 5. *The following equalities are true:*

$$\begin{aligned} 171\widehat{J}_n &= -22\widehat{j}_{n+6} + 123\widehat{j}_{n+5} - 259\widehat{j}_{n+4} + 329\widehat{j}_{n+3} - 193\widehat{j}_{n+2}, \\ 57\widehat{J}_n &= -3\widehat{j}_{n+5} + 9\widehat{j}_{n+4} + 7\widehat{j}_{n+3} - 13\widehat{j}_{n+2} - 22\widehat{j}_{n+1}, \\ 57\widehat{J}_n &= -9\widehat{j}_{n+4} + 46\widehat{j}_{n+3} - 55\widehat{j}_{n+2} - \widehat{j}_{n+1} - 9\widehat{j}_{n}, \\ 57\widehat{J}_n &= -8\widehat{j}_{n+3} + 62\widehat{j}_{n+2} - 127\widehat{j}_{n+1} + 54\widehat{j}_{n} - 27\widehat{j}_{n-1}, \\ 57\widehat{J}_n &= 14\widehat{j}_{n+2} - 23\widehat{j}_{n+1} - 58\widehat{j}_{n} + 29\widehat{j}_{n-1} - 24\widehat{j}_{n-2}, \end{aligned}$$

and

$$\begin{aligned} 198\widehat{j}_n &= 5\widehat{J}_{n+6} - 285\widehat{J}_{n+5} + 1532\widehat{J}_{n+4} - 2764\widehat{J}_{n+3} + 2003\widehat{J}_{n+2}, \\ 66\widehat{j}_n &= -85\widehat{J}_{n+5} + 489\widehat{J}_{n+4} - 898\widehat{J}_{n+3} + 656\widehat{J}_{n+2} + 5\widehat{J}_{n+1}, \\ 22\widehat{j}_n &= -7\widehat{J}_{n+4} + 69\widehat{J}_{n+3} - 178\widehat{J}_{n+2} + 200\widehat{J}_{n+1} - 85\widehat{J}_{n}, \\ 22\widehat{j}_n &= 27\widehat{J}_{n+3} - 87\widehat{J}_{n+2} + 102\widehat{J}_{n+1} - 36\widehat{J}_{n} - 21\widehat{J}_{n-1}, \\ 22\widehat{j}_n &= 75\widehat{J}_{n+2} - 249\widehat{J}_{n+1} + 342\widehat{J}_{n} - 210\widehat{J}_{n-1} + 81\widehat{J}_{n-2}. \end{aligned}$$

Proof. Writing

$$\widehat{J}_n = a \times \widehat{j}_{n+6} + b \times \widehat{j}_{n+5} + c \times \widehat{j}_{n+4} + d \times \widehat{j}_{n+3} + e \times \widehat{j}_{n+2}$$

and solving the system of equations

$$\begin{aligned}
\widehat{J}_0 &= a\times\widehat{j}_6 + b\times\widehat{j}_5 + c\times\widehat{j}_4 + d\times\widehat{j}_3 + e\times\widehat{j}_2\\
\widehat{J}_1 &= a\times\widehat{j}_7 + b\times\widehat{j}_6 + c\times\widehat{j}_5 + d\times\widehat{j}_4 + e\times\widehat{j}_3\\
\widehat{J}_2 &= a\times\widehat{j}_8 + b\times\widehat{j}_7 + c\times\widehat{j}_6 + d\times\widehat{j}_5 + e\times\widehat{j}_4\\
\widehat{J}_3 &= a\times\widehat{j}_9 + b\times\widehat{j}_8 + c\times\widehat{j}_7 + d\times\widehat{j}_6 + e\times\widehat{j}_5\\
\widehat{J}_4 &= a\times\widehat{j}_{10} + b\times\widehat{j}_9 + c\times\widehat{j}_8 + d\times\widehat{j}_7 + e\times\widehat{j}_6
\end{aligned}$$

we find that $a=-\frac{22}{171}, b=\frac{41}{57}, c=-\frac{259}{171}, d=\frac{329}{171}, e=-\frac{193}{171}$. The other equalities can be proved similarly. □

Now, we give a few basic relations between $\{\widehat{J}_n\}$ and $\{\widehat{S}_n\}$.

Lemma 6. *The following equalities are true:*

$$\begin{aligned}
18\widehat{J}_n &= 7\widehat{S}_{n+6} - 39\widehat{S}_{n+5} + 82\widehat{S}_{n+4} - 104\widehat{S}_{n+3} + 61\widehat{S}_{n+2},\\
6\widehat{J}_n &= \widehat{S}_{n+5} - 3\widehat{S}_{n+4} - 2\widehat{S}_{n+3} + 4\widehat{S}_{n+2} + 7\widehat{S}_{n+1},\\
2\widehat{J}_n &= \widehat{S}_{n+4} - 5\widehat{S}_{n+3} + 6\widehat{S}_{n+2} + \widehat{S}_{n},\\
2\widehat{J}_n &= \widehat{S}_{n+3} - 7\widehat{S}_{n+2} + 14\widehat{S}_{n+1} - 6\widehat{S}_{n} + 3\widehat{S}_{n-1},\\
2\widehat{J}_n &= -\widehat{S}_{n+2} + \widehat{S}_{n+1} + 8\widehat{S}_{n} - 4\widehat{S}_{n-1} + 3\widehat{S}_{n-2},
\end{aligned}$$

and

$$\begin{aligned}
99\widehat{S}_n &= \widehat{J}_{n+6} + 42\widehat{J}_{n+5} - 248\widehat{J}_{n+4} + 457\widehat{J}_{n+3} - 332\widehat{J}_{n+2},\\
33\widehat{S}_n &= 16\widehat{J}_{n+5} - 87\widehat{J}_{n+4} + 157\widehat{J}_{n+3} - 113\widehat{J}_{n+2} + \widehat{J}_{n+1},\\
11\widehat{S}_n &= 3\widehat{J}_{n+4} - 17\widehat{J}_{n+3} + 37\widehat{J}_{n+2} - 37\widehat{J}_{n+1} + 16\widehat{J}_{n},\\
11\widehat{S}_n &= \widehat{J}_{n+3} - 2\widehat{J}_{n+2} + 5\widehat{J}_{n+1} - 5\widehat{J}_{n} + 9\widehat{J}_{n-1},\\
11\widehat{S}_n &= 4\widehat{J}_{n+2} - 8\widehat{J}_{n+1} + 9\widehat{J}_{n} + 2\widehat{J}_{n-1} + 3\widehat{J}_{n-2}.
\end{aligned}$$

Next, we present a few basic relations between $\{\widehat{J}_n\}$ and $\{\widehat{R}_n\}$.

Lemma 7. *The following equalities are true:*

$$\begin{aligned}
43245\widehat{J}_n &= 164\widehat{R}_{n+6} + 1182\widehat{R}_{n+5} - 7993\widehat{R}_{n+4} + 15017\widehat{R}_{n+3} - 17692\widehat{R}_{n+2},\\
14415\widehat{J}_n &= 722\widehat{R}_{n+5} - 3375\widehat{R}_{n+4} + 5771\widehat{R}_{n+3} - 6280\widehat{R}_{n+2} + 164\widehat{R}_{n+1},\\
4805\widehat{J}_n &= 319\widehat{R}_{n+4} - 1205\widehat{R}_{n+3} + 1276\widehat{R}_{n+2} - 1630\widehat{R}_{n+1} + 722\widehat{R}_{n},\\
4805\widehat{J}_n &= 709\widehat{R}_{n+3} - 2871\widehat{R}_{n+2} + 2836\widehat{R}_{n+1} - 1511\widehat{R}_{n} + 957\widehat{R}_{n-1},\\
4805\widehat{J}_n &= 1383\widehat{R}_{n+2} - 6381\widehat{R}_{n+1} + 8415\widehat{R}_{n} - 4006\widehat{R}_{n-1} + 2127\widehat{R}_{n-2},
\end{aligned}$$

and

$$\begin{aligned}
396\widehat{R}_n &= 1163\widehat{J}_{n+6} - 7881\widehat{J}_{n+5} + 19664\widehat{J}_{n+4} - 22810\widehat{J}_{n+3} + 10379\widehat{J}_{n+2},\\
132\widehat{R}_n &= -301\widehat{J}_{n+5} + 1515\widehat{J}_{n+4} - 2176\widehat{J}_{n+3} + 746\widehat{J}_{n+2} + 1163\widehat{J}_{n+1},\\
44\widehat{R}_n &= -97\widehat{J}_{n+4} + 579\widehat{J}_{n+3} - 1156\widehat{J}_{n+2} + 1090\widehat{J}_{n+1} - 301\widehat{J}_{n},\\
44\widehat{R}_n &= -3\widehat{J}_{n+3} + 105\widehat{J}_{n+2} - 268\widehat{J}_{n+1} + 378\widehat{J}_{n} - 291\widehat{J}_{n-1},\\
44\widehat{R}_n &= 87\widehat{J}_{n+2} - 229\widehat{J}_{n+1} + 336\widehat{J}_{n} - 270\widehat{J}_{n-1} - 9\widehat{J}_{n-2}.
\end{aligned}$$

Now, we give a few basic relations between $\{\widehat{j}_n\}$ and $\{\widehat{S}_n\}$.

Lemma 8. *The following equalities are true:*

$$36\widehat{j}_n = -83\widehat{S}_{n+6} + 465\widehat{S}_{n+5} - 872\widehat{S}_{n+4} + 706\widehat{S}_{n+3} - 83\widehat{S}_{n+2},$$

$$
\begin{aligned}
12\widehat{j}_n &= -11\widehat{S}_{n+5} + 69\widehat{S}_{n+4} - 152\widehat{S}_{n+3} + 166\widehat{S}_{n+2} - 83\widehat{S}_{n+1}, \\
4\widehat{j}_n &= \widehat{S}_{n+4} - 3\widehat{S}_{n+3} + 4\widehat{S}_{n+2} - 2\widehat{S}_{n+1} - 11\widehat{S}_n, \\
4\widehat{j}_n &= 3\widehat{S}_{n+3} - 9\widehat{S}_{n+2} + 12\widehat{S}_{n+1} - 18\widehat{S}_n + 3\widehat{S}_{n-1}, \\
4\widehat{j}_n &= 9\widehat{S}_{n+2} - 27\widehat{S}_{n+1} + 24\widehat{S}_n - 18\widehat{S}_{n-1} + 9\widehat{S}_{n-2},
\end{aligned}
$$

and

$$
\begin{aligned}
171\widehat{S}_n &= -44\widehat{j}_{n+6} + 246\widehat{j}_{n+5} - 461\widehat{j}_{n+4} + 373\widehat{j}_{n+3} - 44\widehat{j}_{n+2}, \\
57\widehat{S}_n &= -6\widehat{j}_{n+5} + 37\widehat{j}_{n+4} - 81\widehat{j}_{n+3} + 88\widehat{j}_{n+2} - 44\widehat{j}_{n+1}, \\
57\widehat{S}_n &= \widehat{j}_{n+4} - 3\widehat{j}_{n+3} + 4\widehat{j}_{n+2} - 2\widehat{j}_{n+1} - 18\widehat{j}_n, \\
57\widehat{S}_n &= 3\widehat{j}_{n+3} - 9\widehat{j}_{n+2} + 12\widehat{j}_{n+1} - 25\widehat{j}_n + 3\widehat{j}_{n-1}, \\
57\widehat{S}_n &= 9\widehat{j}_{n+2} - 27\widehat{j}_{n+1} + 17\widehat{j}_n - 18\widehat{j}_{n-1} + 9\widehat{j}_{n-2}.
\end{aligned}
$$

Next, we present a few basic relations between $\{\widehat{j}_n\}$ and $\{\widehat{R}_n\}$.

Lemma 9. *The following equalities are true:*

$$
\begin{aligned}
43245\widehat{j}_n &= 11881\widehat{R}_{n+6} - 71634\widehat{R}_{n+5} + 151312\widehat{R}_{n+4} - 141779\widehat{R}_{n+3} + 41176\widehat{R}_{n+2}, \\
14415\widehat{j}_n &= -116\widehat{R}_{n+5} - 1047\widehat{R}_{n+4} + 8185\widehat{R}_{n+3} - 13997\widehat{R}_{n+2} + 11881\widehat{R}_{n+1}, \\
4805\widehat{j}_n &= -581\widehat{R}_{n+4} + 3231\widehat{R}_{n+3} - 5207\widehat{R}_{n+2} + 4231\widehat{R}_{n+1} - 116\widehat{R}_n, \\
4805\widehat{j}_n &= -255\widehat{R}_{n+3} + 2346\widehat{R}_{n+2} - 3903\widehat{R}_{n+1} + 3951\widehat{R}_n - 1743\widehat{R}_{n-1}, \\
4805\widehat{j}_n &= 816\widehat{R}_{n+2} - 588\widehat{R}_{n+1} + 381\widehat{R}_n + 42\widehat{R}_{n-1} - 765\widehat{R}_{n-2},
\end{aligned}
$$

and

$$
\begin{aligned}
342\widehat{R}_n &= 457\widehat{j}_{n+6} - 2397\widehat{j}_{n+5} + 3994\widehat{j}_{n+4} - 2396\widehat{j}_{n+3} - 1025\widehat{j}_{n+2}, \\
114\widehat{R}_n &= 115\widehat{j}_{n+5} - 649\widehat{j}_{n+4} + 1334\widehat{j}_{n+3} - 1408\widehat{j}_{n+2} + 457\widehat{j}_{n+1}, \\
114\widehat{R}_n &= 41\widehat{j}_{n+4} - 161\widehat{j}_{n+3} + 202\widehat{j}_{n+2} - 348\widehat{j}_{n+1} + 345\widehat{j}_n, \\
114\widehat{R}_n &= 85\widehat{j}_{n+3} - 331\widehat{j}_{n+2} + 226\widehat{j}_{n+1} + 58\widehat{j}_n + 123\widehat{j}_{n-1}, \\
114\widehat{R}_n &= 179\widehat{j}_{n+2} - 879\widehat{j}_{n+1} + 1248\widehat{j}_n - 472\widehat{j}_{n-1} + 255\widehat{j}_{n-2}.
\end{aligned}
$$

Now, we give a few basic relations between $\{\widehat{S}_n\}$ and $\{\widehat{R}_n\}$.

Lemma 10. *The following equalities are true:*

$$
\begin{aligned}
43245\widehat{S}_n &= -3857\widehat{R}_{n+6} + 23568\widehat{R}_{n+5} - 50024\widehat{R}_{n+4} + 47053\widehat{R}_{n+3} - 13622\widehat{R}_{n+2}, \\
14415\widehat{S}_n &= 142\widehat{R}_{n+5} + 39\widehat{R}_{n+4} - 2315\widehat{R}_{n+3} + 4459\widehat{R}_{n+2} - 3857\widehat{R}_{n+1}, \\
4805\widehat{S}_n &= 297\widehat{R}_{n+4} - 1387\widehat{R}_{n+3} + 2149\widehat{R}_{n+2} - 1617\widehat{R}_{n+1} + 142\widehat{R}_n, \\
4805\widehat{S}_n &= 395\widehat{R}_{n+3} - 1712\widehat{R}_{n+2} + 2541\widehat{R}_{n+1} - 1937\widehat{R}_n + 891\widehat{R}_{n-1}, \\
4805\widehat{S}_n &= 658\widehat{R}_{n+2} - 2594\widehat{R}_{n+1} + 3593\widehat{R}_n - 1874\widehat{R}_{n-1} + 1185\widehat{R}_{n-2},
\end{aligned}
$$

and

$$
\begin{aligned}
72\widehat{R}_n &= -287\widehat{S}_{n+6} + 1509\widehat{S}_{n+5} - 2516\widehat{S}_{n+4} + 1510\widehat{S}_{n+3} + 649\widehat{S}_{n+2}, \\
24\widehat{R}_n &= -71\widehat{S}_{n+5} + 405\widehat{S}_{n+4} - 836\widehat{S}_{n+3} + 886\widehat{S}_{n+2} - 287\widehat{S}_{n+1}, \\
8\widehat{R}_n &= -7\widehat{S}_{n+4} + 29\widehat{S}_{n+3} - 36\widehat{S}_{n+2} + 70\widehat{S}_{n+1} - 71\widehat{S}_n, \\
8\widehat{R}_n &= -13\widehat{S}_{n+3} + 55\widehat{S}_{n+2} - 28\widehat{S}_{n+1} - 22\widehat{S}_n - 21\widehat{S}_{n-1}, \\
8\widehat{R}_n &= -23\widehat{S}_{n+2} + 141\widehat{S}_{n+1} - 204\widehat{S}_n + 70\widehat{S}_{n-1} - 39\widehat{S}_{n-2}.
\end{aligned}
$$

6. On the recurrence properties of binomial transform of the generalized fifth order Jacobsthal sequence

Taking $r_1 = 6, s_1 = -13, t_1 = 14, u_1 = -7, v_1 = 3$ and $H_n = \widehat{R}_n$ in Theorem 3, we obtain the following Proposition.

Proposition 2. *For $n \in \mathbb{Z}$, binomial Transform of the generalized fifth order Jacobsthal sequence have the following identity:*

$$\begin{aligned} b_{-n} &= \frac{1}{24}3^{-n}(b_0\widehat{R}_n^4 - 4b_n\widehat{R}_n^3 + 3b_0\widehat{R}_{2n}^2 + 12\widehat{R}_n^2 b_{2n} - 6b_0\widehat{R}_n^2\widehat{R}_{2n} - 6b_0\widehat{R}_{4n} - 8b_n\widehat{R}_{3n} - 12\widehat{R}_{2n}b_{2n} - 24\widehat{R}_n b_{3n} + 24b_{4n} \\ &\quad +8b_0\widehat{R}_n\widehat{R}_{3n} + 12b_n\widehat{R}_n\widehat{R}_{2n}) \\ &= 3^{-n}(b_{4n} - \widehat{R}_n b_{3n} + \frac{1}{2}(\widehat{R}_n^2 - \widehat{R}_{2n})b_{2n} - \frac{1}{6}(\widehat{R}_n^3 + 2\widehat{R}_{3n} - 3\widehat{R}_{2n}\widehat{R}_n)b_n \\ &\quad +\frac{1}{24}(\widehat{R}_n^4 + 3\widehat{R}_{2n}^2 - 6\widehat{R}_n^2\widehat{R}_{2n} - 6\widehat{R}_{4n} + 8\widehat{R}_{3n}\widehat{R}_n)b_0). \end{aligned}$$

Using Proposition 2 (and Corollary 2), we obtain the following corollary which gives the connection between the special cases of binomial transform of generalized fifth order Jacobsthal sequence at the positive index and the negative index: for binomial transform of fifth order Jacobsthal, fifth order Jacobsthal-Lucas numbers: take $b_n = \widehat{J}_n$ with $\widehat{J}_0 = 0, \widehat{J}_1 = 1, \widehat{J}_2 = 3, \widehat{J}_3 = 7, \widehat{J}_4 = 15$, take $b_n = \widehat{j}_n$ with $\widehat{j}_0 = 2, \widehat{j}_1 = 3, \widehat{j}_2 = 9, \widehat{j}_3 = 30, \widehat{j}_4 = 96$, take $b_n = \widehat{S}_n$ with $\widehat{S}_0 = 0, \widehat{S}_1 = 1, \widehat{S}_2 = 3, \widehat{S}_3 = 8, \widehat{S}_4 = 22$, take $b_n = \widehat{R}_n$ with $\widehat{R}_0 = 5, \widehat{R}_1 = 6, \widehat{R}_2 = 10, \widehat{R}_3 = 24, \widehat{R}_4 = 70$,respectively. Note that in this case we have $H_n = \widehat{R}_n$. Note also that $G_n \neq \widehat{S}_n$.

Corollary 5. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a) *Recurrence relations of binomial transforms of fifth order Jacobsthal numbers (take $b_n = \widehat{J}_n$ in Proposition 2):*

$$\widehat{J}_{-n} = 3^{-n}(\widehat{J}_{4n} - \widehat{R}_n\widehat{J}_{3n} + \frac{1}{2}(\widehat{R}_n^2 - \widehat{R}_{2n})\widehat{J}_{2n} - \frac{1}{6}(\widehat{R}_n^3 + 2\widehat{R}_{3n} - 3\widehat{R}_{2n}\widehat{R}_n)\widehat{J}_n).$$

(b) *Recurrence relations of binomial transforms of fifth order Jacobsthal-Lucas numbers (take $b_n = \widehat{j}_n$ in Proposition 2):*

$$\widehat{j}_{-n} = 3^{-n}(\widehat{j}_{4n} - \widehat{R}_n\widehat{j}_{3n} + \frac{1}{2}(\widehat{R}_n^2 - \widehat{R}_{2n})\widehat{j}_{2n} - \frac{1}{6}(\widehat{R}_n^3 + 2\widehat{R}_{3n} - 3\widehat{R}_{2n}\widehat{R}_n)\widehat{j}_n + \frac{1}{12}(\widehat{R}_n^4 + 3\widehat{R}_{2n}^2 - 6\widehat{R}_n^2\widehat{R}_{2n} - 6\widehat{R}_{4n} + 8\widehat{R}_{3n}\widehat{R}_n)).$$

(c) *Recurrence relations of binomial transforms of adjusted fifth order Jacobsthal numbers (take $b_n = \widehat{S}_n$ in Proposition 2):*

$$\widehat{S}_{-n} = 3^{-n}(\widehat{S}_{4n} - \widehat{R}_n\widehat{S}_{3n} + \frac{1}{2}(\widehat{R}_n^2 - \widehat{R}_{2n})\widehat{S}_{2n} - \frac{1}{6}(\widehat{R}_n^3 + 2\widehat{R}_{3n} - 3\widehat{R}_{2n}\widehat{R}_n)\widehat{S}_n).$$

(d) *Recurrence relations of binomial transforms of modified fifth order Jacobsthal-Lucas numbers (take $b_n = \widehat{R}_n$ in Proposition 2 or take $H_n = \widehat{R}_n$ in Corollary 2):*

$$\widehat{R}_{-n} = \frac{1}{24}3^{-n}(\widehat{R}_n^4 + 3\widehat{R}_{2n}^2 - 6\widehat{R}_n^2\widehat{R}_{2n} - 6\widehat{R}_{4n} + 8\widehat{R}_{3n}\widehat{R}_n).$$

7. Sum formulas

7.1. Sums of terms with positive subscripts

The following proposition presents some formulas of binomial transform of generalized fifth order Jacobsthal numbers with positive subscripts.

Proposition 3. *If $r = 6, s = -13, t = 14, u = -7, v = 3$, then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} b_k = \frac{1}{2}(b_{n+5} - 5b_{n+4} + 8b_{n+3} - 6b_{n+2} + b_{n+1} - b_4 + 5b_3 - 8b_2 + 6b_1 - b_0)$.
(b) $\sum_{k=0}^{n} b_{2k} = \frac{1}{88}(21b_{2n+2} - 103b_{2n+1} + 244b_{2n} - 98b_{2n-1} + 69b_{2n-2} - 21b_4 + 103b_3 - 156b_2 + 98b_1 + 19b_0)$.
(c) $\sum_{k=0}^{n} b_{2k+1} = \frac{1}{88}(23b_{2n+2} - 29b_{2n+1} + 196b_{2n} - 78b_{2n-1} + 63b_{2n-2} - 23b_4 + 117b_3 - 196b_2 + 166b_1 - 63b_0)$.

Proof. Take $r = 6, s = -13, t = 14, u = -7, v = 3$, in Theorem 2.1 in [28]. □

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of fifth order Jacobsthal numbers (take $b_n = \widehat{J}_n$ with $\widehat{J}_0 = 0, \widehat{J}_1 = 1, \widehat{J}_2 = 3, \widehat{J}_3 = 7, \widehat{J}_4 = 15$).

Corollary 6. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} \widehat{J}_k = \frac{1}{2}(\widehat{J}_{n+5} - 5\widehat{J}_{n+4} + 8\widehat{J}_{n+3} - 6\widehat{J}_{n+2} + \widehat{J}_{n+1} + 2)$.
(b) $\sum_{k=0}^{n} \widehat{J}_{2k} = \frac{1}{88}(21\widehat{J}_{2n+2} - 103\widehat{J}_{2n+1} + 244\widehat{J}_{2n} - 98\widehat{J}_{2n-1} + 69\widehat{J}_{2n-2} + 36)$.
(c) $\sum_{k=0}^{n} \widehat{J}_{2k+1} = \frac{1}{88}(23\widehat{J}_{2n+2} - 29\widehat{J}_{2n+1} + 196\widehat{J}_{2n} - 78\widehat{J}_{2n-1} + 63\widehat{J}_{2n-2} + 52)$.

Taking $b_n = \widehat{j}_n$ with $\widehat{j}_0 = 2, \widehat{j}_1 = 3, \widehat{j}_2 = 9, \widehat{j}_3 = 30, \widehat{j}_4 = 96$ in the last proposition, we have the following corollary which presents sum formulas of binomial transform of fifth order Jacobsthal-Lucas numbers.

Corollary 7. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} \widehat{j}_k = \frac{1}{2}(\widehat{j}_{n+5} - 5\widehat{j}_{n+4} + 8\widehat{j}_{n+3} - 6\widehat{j}_{n+2} + \widehat{j}_{n+1} - 2)$.
(b) $\sum_{k=0}^{n} \widehat{j}_{2k} = \frac{1}{88}(21\widehat{j}_{2n+2} - 103\widehat{j}_{2n+1} + 244\widehat{j}_{2n} - 98\widehat{j}_{2n-1} + 69\widehat{j}_{2n-2} + 2)$.
(c) $\sum_{k=0}^{n} \widehat{j}_{2k+1} = \frac{1}{88}(23\widehat{j}_{2n+2} - 29\widehat{j}_{2n+1} + 196\widehat{j}_{2n} - 78\widehat{j}_{2n-1} + 63\widehat{j}_{2n-2} - 90)$.

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of adjusted fifth order Jacobsthal numbers (take $b_n = \widehat{S}_n$ with $\widehat{S}_0 = 0, \widehat{S}_1 = 1, \widehat{S}_2 = 3, \widehat{S}_3 = 8, \widehat{S}_4 = 22$).

Corollary 8. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} \widehat{S}_k = \frac{1}{2}(\widehat{S}_{n+5} - 5\widehat{S}_{n+4} + 8\widehat{S}_{n+3} - 6\widehat{S}_{n+2} + \widehat{S}_{n+1})$.
(b) $\sum_{k=0}^{n} \widehat{S}_{2k} = \frac{1}{88}(21\widehat{S}_{2n+2} - 103\widehat{S}_{2n+1} + 244\widehat{S}_{2n} - 98\widehat{S}_{2n-1} + 69\widehat{S}_{2n-2} - 8)$.
(c) $\sum_{k=0}^{n} \widehat{S}_{2k+1} = \frac{1}{88}(23\widehat{S}_{2n+2} - 29\widehat{S}_{2n+1} + 196\widehat{S}_{2n} - 78\widehat{S}_{2n-1} + 63\widehat{S}_{2n-2} + 8)$.

Taking $b_n = \widehat{R}_n$ with $\widehat{R}_0 = 5, \widehat{R}_1 = 6, \widehat{R}_2 = 10, \widehat{R}_3 = 24, \widehat{R}_4 = 70$ in the last proposition, we have the following corollary which presents sum formulas of binomial transform of modified fifth order Jacobsthal-Lucas numbers.

Corollary 9. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} \widehat{R}_k = \frac{1}{2}(\widehat{R}_{n+5} - 5\widehat{R}_{n+4} + 8\widehat{R}_{n+3} - 6\widehat{R}_{n+2} + \widehat{R}_{n+1} + 1)$.
(b) $\sum_{k=0}^{n} \widehat{R}_{2k} = \frac{1}{88}(21\widehat{R}_{2n+2} - 103\widehat{R}_{2n+1} + 244\widehat{R}_{2n} - 98\widehat{R}_{2n-1} + 69\widehat{R}_{2n-2} + 125)$.
(c) $\sum_{k=0}^{n} \widehat{R}_{2k+1} = \frac{1}{88}(23\widehat{R}_{2n+2} - 29\widehat{R}_{2n+1} + 196\widehat{R}_{2n} - 78\widehat{R}_{2n-1} + 63\widehat{R}_{2n-2} - 81)$.

7.2. Sums of terms with negative subscripts

The following proposition presents some formulas of binomial transform of generalized fifth order Jacobsthal numbers with negative subscripts.

Proposition 4. *If $r = 6, s = -13, t = 14, u = -7, v = 3$, then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} b_{-k} = \frac{1}{2}(-b_{-n+4} + 5b_{-n+3} - 8b_{-n+2} + 6b_{-n+1} - b_{-n} + b_4 - 5b_3 + 8b_2 - 6b_1 + b_0)$.
(b) $\sum_{k=1}^{n} b_{-2k} = \frac{1}{88}(-23b_{-2n+3} + 117b_{-2n+2} - 196b_{-2n+1} + 166b_{-2n} - 63b_{-2n-1} + 21b_4 - 103b_3 + 156b_2 - 98b_1 - 19b_0)$.
(c) $\sum_{k=1}^{n} b_{-2k+1} = \frac{1}{88}(-21b_{-2n+3} + 103b_{-2n+2} - 156b_{-2n+1} + 98b_{-2n} - 69b_{-2n-1} + 23b_4 - 117b_3 + 196b_2 - 166b_1 + 63b_0)$.

Proof. Take $r = 6, s = -13, t = 14, u = -7, v = 3$, in Theorem 3.1 in [28]. □

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of fifth order Jacobsthal numbers (take $b_n = \widehat{J}_n$ with $\widehat{J}_0 = 0, \widehat{J}_1 = 1, \widehat{J}_2 = 3, \widehat{J}_3 = 7, \widehat{J}_4 = 15$).

Corollary 10. *For $n \geq 1$, binomial transform of fifth order Jacobsthal numbers have the following properties.*

(a) $\sum_{k=1}^{n} \widehat{J}_{-k} = \frac{1}{2}(-\widehat{J}_{-n+4} + 5\widehat{J}_{-n+3} - 8\widehat{J}_{-n+2} + 6\widehat{J}_{-n+1} - \widehat{J}_{-n} - 2)$.

(b) $\sum_{k=1}^{n}\widehat{J}_{-2k}=\frac{1}{88}(-23\widehat{J}_{-2n+3}+117\widehat{J}_{-2n+2}-196\widehat{J}_{-2n+1}+166\widehat{J}_{-2n}-63\widehat{J}_{-2n-1}-36)$.
(c) $\sum_{k=1}^{n}\widehat{J}_{-2k+1}=\frac{1}{88}(-21\widehat{J}_{-2n+3}+103\widehat{J}_{-2n+2}-156\widehat{J}_{-2n+1}+98\widehat{J}_{-2n}-69\widehat{J}_{-2n-1}-52)$.

Taking $b_n=\widehat{j}_n$ with $\widehat{j}_0=2,\widehat{j}_1=3,\widehat{j}_2=9,\widehat{j}_3=30,\widehat{j}_4=96$ in the last proposition, we have the following corollary which presents sum formulas of binomial transform of fifth order Jacobsthal-Lucas numbers.

Corollary 11. *For $n\geq 1$, binomial transform of fifth order Jacobsthal-Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n}\widehat{j}_{-k}=\frac{1}{2}(-\widehat{j}_{-n+4}+5\widehat{j}_{-n+3}-8\widehat{j}_{-n+2}+6\widehat{j}_{-n+1}-\widehat{j}_{-n}+2)$.
(b) $\sum_{k=1}^{n}\widehat{j}_{-2k}=\frac{1}{88}(-23\widehat{j}_{-2n+3}+117\widehat{j}_{-2n+2}-196\widehat{j}_{-2n+1}+166\widehat{j}_{-2n}-63\widehat{j}_{-2n-1}-2)$.
(c) $\sum_{k=1}^{n}\widehat{j}_{-2k+1}=\frac{1}{88}(-21\widehat{j}_{-2n+3}+103\widehat{j}_{-2n+2}-156\widehat{j}_{-2n+1}+98\widehat{j}_{-2n}-69\widehat{j}_{-2n-1}+90)$.

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of adjusted fifth order Jacobsthal numbers (take $b_n=\widehat{S}_n$ with $\widehat{S}_0=0,\widehat{S}_1=1,\widehat{S}_2=3,\widehat{S}_3=8,\widehat{S}_4=22$).

Corollary 12. *For $n\geq 1$, binomial transform of adjusted fifth order Jacobsthal numbers have the following properties.*

(a) $\sum_{k=1}^{n}\widehat{S}_{-k}=\frac{1}{2}(-\widehat{S}_{-n+4}+5\widehat{S}_{-n+3}-8\widehat{S}_{-n+2}+6\widehat{S}_{-n+1}-\widehat{S}_{-n})$.
(b) $\sum_{k=1}^{n}\widehat{S}_{-2k}=\frac{1}{88}(-23\widehat{S}_{-2n+3}+117\widehat{S}_{-2n+2}-196\widehat{S}_{-2n+1}+166\widehat{S}_{-2n}-63\widehat{S}_{-2n-1}+8)$.
(c) $\sum_{k=1}^{n}\widehat{S}_{-2k+1}=\frac{1}{88}(-21\widehat{S}_{-2n+3}+103\widehat{S}_{-2n+2}-156\widehat{S}_{-2n+1}+98\widehat{S}_{-2n}-69\widehat{S}_{-2n-1}-8)$.

Taking $b_n=\widehat{R}_n$ with $\widehat{R}_0=5,\widehat{R}_1=6,\widehat{R}_2=10,\widehat{R}_3=24,\widehat{R}_4=70$ in the last proposition, we have the following corollary which presents sum formulas of binomial transform of modified fifth order Jacobsthal-Lucas numbers.

Corollary 13. *For $n\geq 1$, binomial transform of modified fifth order Jacobsthal-Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n}\widehat{R}_{-k}=\frac{1}{2}(-\widehat{R}_{-n+4}+5\widehat{R}_{-n+3}-8\widehat{R}_{-n+2}+6\widehat{R}_{-n+1}-\widehat{R}_{-n}-1)$.
(b) $\sum_{k=1}^{n}\widehat{R}_{-2k}=\frac{1}{88}(-23\widehat{R}_{-2n+3}+117\widehat{R}_{-2n+2}-196\widehat{R}_{-2n+1}+166\widehat{R}_{-2n}-63\widehat{R}_{-2n-1}-125)$.
(c) $\sum_{k=1}^{n}\widehat{R}_{-2k+1}=\frac{1}{88}(-21\widehat{R}_{-2n+3}+103\widehat{R}_{-2n+2}-156\widehat{R}_{-2n+1}+98\widehat{R}_{-2n}-69\widehat{R}_{-2n-1}+81)$.

8. Matrices related with binomial transform of generalized fifth order Jacobsthal numbers

We define the square matrix A of order 5 as:

$$A=\begin{pmatrix}6&-13&14&-7&3\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{pmatrix},$$

such that $\det A=3$. From (1) we have

$$\begin{pmatrix}b_{n+4}\\b_{n+3}\\b_{n+2}\\b_{n+1}\\b_n\end{pmatrix}=\begin{pmatrix}6&-13&14&-7&3\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{pmatrix}\begin{pmatrix}b_{n+3}\\b_{n+2}\\b_{n+1}\\b_n\\b_{n-1}\end{pmatrix}, \tag{19}$$

and from (6) (or using (19) and induction) we have

$$\begin{pmatrix}b_{n+4}\\b_{n+3}\\b_{n+2}\\b_{n+1}\\b_n\end{pmatrix}=\begin{pmatrix}6&-13&14&-7&3\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{pmatrix}^n\begin{pmatrix}b_4\\b_3\\b_2\\b_1\\b_0\end{pmatrix}.$$

If we take $b_n = \widehat{S}_n$ in (19) we have

$$\begin{pmatrix} \widehat{S}_{n+4} \\ \widehat{S}_{n+3} \\ \widehat{S}_{n+2} \\ \widehat{S}_{n+1} \\ \widehat{S}_{n} \end{pmatrix} = \begin{pmatrix} 6 & -13 & 14 & -7 & 3 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \widehat{S}_{n+3} \\ \widehat{S}_{n+2} \\ \widehat{S}_{n+1} \\ \widehat{S}_{n} \\ \widehat{S}_{n-1} \end{pmatrix}. \tag{20}$$

We also, for $n \geq 0$, define

$$B_n = \begin{pmatrix} \sum_{k=0}^{n+1}\sum_{l=k}^{n+1}\sum_{p=l}^{n+1}\widehat{S}_k & E_1 & E_6 & E_{11} & 3\sum_{k=0}^{n}\sum_{l=k}^{n}\sum_{p=l}^{n}\widehat{S}_k \\ \sum_{k=0}^{n}\sum_{l=k}^{n}\sum_{p=l}^{n}\widehat{S}_k & E_2 & E_7 & E_{12} & 3\sum_{k=0}^{n-1}\sum_{l=k}^{n-1}\sum_{p=l}^{n-1}\widehat{S}_k \\ \sum_{k=0}^{n-1}\sum_{l=k}^{n-1}\sum_{p=l}^{n-1}\widehat{S}_k & E_3 & E_8 & E_{13} & 3\sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\sum_{p=l}^{n-2}\widehat{S}_k \\ \sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\sum_{p=l}^{n-2}\widehat{S}_k & E_4 & E_9 & E_{14} & 3\sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\sum_{p=l}^{n-3}\widehat{S}_k \\ \sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\sum_{p=l}^{n-3}\widehat{S}_k & E_5 & E_{10} & E_{15} & 3\sum_{k=0}^{n-4}\sum_{l=k}^{n-4}\sum_{p=l}^{n-4}\widehat{S}_k \end{pmatrix},$$

and

$$C_n = \begin{pmatrix} b_{n+1} & -13b_n+14b_{n-1}-7b_{n-2}+3b_{n-3} & 14b_n-7b_{n-1}+3b_{n-2} & -7b_n+3b_{n-1} & 3b_n \\ b_n & -13b_{n-1}+14b_{n-2}-7b_{n-3}+3b_{n-4} & 14b_{n-1}-7b_{n-2}+3b_{n-3} & -7b_{n-1}+3b_{n-2} & 3b_{n-1} \\ b_{n-1} & -13b_{n-2}+14b_{n-3}-7b_{n-4}+3b_{n-5} & 14b_{n-2}-7b_{n-3}+3b_{n-4} & -7b_{n-2}+3b_{n-3} & 3b_{n-2} \\ b_{n-2} & -13b_{n-3}+14b_{n-4}-7b_{n-5}+3b_{n-6} & 14b_{n-3}-7b_{n-4}+3b_{n-5} & -7b_{n-3}+3b_{n-4} & 3b_{n-3} \\ b_{n-3} & -13b_{n-4}+14b_{n-5}-7b_{n-6}+3b_{n-7} & 14b_{n-4}-7b_{n-5}+3b_{n-6} & -7b_{n-4}+3b_{n-5} & 3b_{n-4} \end{pmatrix},$$

where

$$\begin{Bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \\ E_7 \\ E_8 \\ E_9 \\ E_{10} \\ E_{11} \\ E_{12} \\ E_{13} \\ E_{14} \\ E_{15} \end{Bmatrix} = \begin{Bmatrix} -13\sum_{k=0}^{n}\sum_{l=k}^{n}\sum_{p=l}^{n}\widehat{S}_k+14\sum_{k=0}^{n-1}\sum_{l=k}^{n-1}\sum_{p=l}^{n-1}\widehat{S}_k-7\sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\sum_{p=l}^{n-2}\widehat{S}_k+3\sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\sum_{p=l}^{n-3}\widehat{S}_k \\ -13\sum_{k=0}^{n-1}\sum_{l=k}^{n-1}\sum_{p=l}^{n-1}\widehat{S}_k+14\sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\sum_{p=l}^{n-2}\widehat{S}_k-7\sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\sum_{p=l}^{n-3}\widehat{S}_k+3\sum_{k=0}^{n-4}\sum_{l=k}^{n-4}\sum_{p=l}^{n-4}\widehat{S}_k \\ -13\sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\sum_{p=l}^{n-2}\widehat{S}_k+14\sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\sum_{p=l}^{n-3}\widehat{S}_k-7\sum_{k=0}^{n-4}\sum_{l=k}^{n-4}\sum_{p=l}^{n-4}\widehat{S}_k+3\sum_{k=0}^{n-5}\sum_{l=k}^{n-5}\sum_{p=l}^{n-5}\widehat{S}_k \\ -13\sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\sum_{p=l}^{n-3}\widehat{S}_k+14\sum_{k=0}^{n-4}\sum_{l=k}^{n-4}\sum_{p=l}^{n-4}\widehat{S}_k-7\sum_{k=0}^{n-5}\sum_{l=k}^{n-5}\sum_{p=l}^{n-5}\widehat{S}_k+3\sum_{k=0}^{n-6}\sum_{l=k}^{n-6}\sum_{p=l}^{n-6}\widehat{S}_k \\ -13\sum_{k=0}^{n-4}\sum_{l=k}^{n-4}\sum_{p=l}^{n-4}\widehat{S}_k+14\sum_{k=0}^{n-5}\sum_{l=k}^{n-5}\sum_{p=l}^{n-5}\widehat{S}_k-7\sum_{k=0}^{n-6}\sum_{l=k}^{n-6}\sum_{p=l}^{n-6}\widehat{S}_k+3\sum_{k=0}^{n-7}\sum_{l=k}^{n-7}\sum_{p=l}^{n-7}\widehat{S}_k \\ 14\sum_{k=0}^{n}\sum_{l=k}^{n}\sum_{p=l}^{n}\widehat{S}_k-7\sum_{k=0}^{n-1}\sum_{l=k}^{n-1}\sum_{p=l}^{n-1}\widehat{S}_k+3\sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\sum_{p=l}^{n-2}\widehat{S}_k \\ 14\sum_{k=0}^{n-1}\sum_{l=k}^{n-1}\sum_{p=l}^{n-1}\widehat{S}_k-7\sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\sum_{p=l}^{n-2}\widehat{S}_k+3\sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\sum_{p=l}^{n-3}\widehat{S}_k \\ 14\sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\sum_{p=l}^{n-2}\widehat{S}_k-7\sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\sum_{p=l}^{n-3}\widehat{S}_k+3\sum_{k=0}^{n-4}\sum_{l=k}^{n-4}\sum_{p=l}^{n-4}\widehat{S}_k \\ 14\sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\sum_{p=l}^{n-3}\widehat{S}_k-7\sum_{k=0}^{n-4}\sum_{l=k}^{n-4}\sum_{p=l}^{n-4}\widehat{S}_k+3\sum_{k=0}^{n-5}\sum_{l=k}^{n-5}\sum_{p=l}^{n-5}\widehat{S}_k \\ 14\sum_{k=0}^{n-4}\sum_{l=k}^{n-4}\sum_{p=l}^{n-4}\widehat{S}_k-7\sum_{k=0}^{n-5}\sum_{l=k}^{n-5}\sum_{p=l}^{n-5}\widehat{S}_k+3\sum_{k=0}^{n-6}\sum_{l=k}^{n-6}\sum_{p=l}^{n-6}\widehat{S}_k \\ -7\sum_{k=0}^{n}\sum_{l=k}^{n}\sum_{p=l}^{n}\widehat{S}_k+3\sum_{k=0}^{n-1}\sum_{l=k}^{n-1}\sum_{p=l}^{n-1}\widehat{S}_k \\ -7\sum_{k=0}^{n-1}\sum_{l=k}^{n-1}\sum_{p=l}^{n-1}\widehat{S}_k+3\sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\sum_{p=l}^{n-2}\widehat{S}_k \\ -7\sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\sum_{p=l}^{n-2}\widehat{S}_k+3\sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\sum_{p=l}^{n-3}\widehat{S}_k \\ -7\sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\sum_{p=l}^{n-3}\widehat{S}_k+3\sum_{k=0}^{n-4}\sum_{l=k}^{n-4}\sum_{p=l}^{n-4}\widehat{S}_k \\ -7\sum_{k=0}^{n-4}\sum_{l=k}^{n-4}\sum_{p=l}^{n-4}\widehat{S}_k+3\sum_{k=0}^{n-5}\sum_{l=k}^{n-5}\sum_{p=l}^{n-5}\widehat{S}_k \end{Bmatrix}.$$

By convention, we assume that

$$\sum_{k=0}^{0}\sum_{l=k}^{0}\sum_{p=l}^{0}\widehat{S}_k = 0, \quad \sum_{k=0}^{-1}\sum_{l=k}^{-1}\sum_{p=l}^{-1}\widehat{S}_k = 0, \quad \sum_{k=0}^{-2}\sum_{l=k}^{-2}\sum_{p=l}^{-2}\widehat{S}_k = 0, \quad \sum_{k=0}^{-3}\sum_{l=k}^{-3}\sum_{p=l}^{-3}\widehat{S}_k = 0,$$

$$\sum_{k=0}^{-4}\sum_{l=k}^{-4}\sum_{p=l}^{-4}\widehat{S}_k = \frac{1}{3}, \quad \sum_{k=0}^{-5}\sum_{l=k}^{-5}\sum_{p=l}^{-5}\widehat{S}_k = \frac{7}{9}, \quad \sum_{k=0}^{-6}\sum_{l=k}^{-6}\sum_{p=l}^{-6}\widehat{S}_k = \frac{7}{27}, \quad \sum_{k=0}^{-7}\sum_{l=k}^{-7}\sum_{p=l}^{-7}\widehat{S}_k = -\frac{128}{81}.$$

Theorem 6. *For all integers $m, n \geq 0$, we have*

(a) $B_n = A^n$.
(b) $C_1 A^n = A^n C_1$.

(c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof. **(a)** Proof can be done by mathematical induction on n.
(b) After matrix multiplication, (b) follows.
(c) We have $C_n = AC_{n-1}$. From the last equation, using induction, we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^m C_1 = A^{n-1}C_1 A^m = C_n B_m$$

and similarly

$$C_{n+m} = B_m C_n.$$

□

Theorem 7. *For $m, n \geq 0$, we have*

$$\begin{aligned} b_{n+m} &= b_n \sum_{k=0}^{m+1}\sum_{l=k}^{m+1}\sum_{p=l}^{m+1} \widehat{S}_k \\ &+b_{n-1}\left(-13\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{S}_k + 14\sum_{k=0}^{m-1}\sum_{l=k}^{m-1}\sum_{p=l}^{m-1}\widehat{S}_k - 7\sum_{k=0}^{m-2}\sum_{l=k}^{m-2}\sum_{p=l}^{m-2}\widehat{S}_k + 3\sum_{k=0}^{m-3}\sum_{l=k}^{m-3}\sum_{p=l}^{m-3}\widehat{S}_k\right) \\ &+b_{n-2}\left(14\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{S}_k - 7\sum_{k=0}^{m-1}\sum_{l=k}^{m-1}\sum_{p=l}^{m-1}\widehat{S}_k + 3\sum_{k=0}^{m-2}\sum_{l=k}^{m-2}\sum_{p=l}^{m-2}\widehat{S}_k\right) \\ &+b_{n-3}\left(-7\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{S}_k + 3\sum_{k=0}^{m-1}\sum_{l=k}^{m-1}\sum_{p=l}^{m-1}\widehat{S}_k\right) + 3b_{n-4}\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{S}_k. \end{aligned}$$

Proof. From the equation $C_{n+m} = C_n B_m = B_m C_n$, we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation, we say that an element of C_{n+m} is the product of a row C_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and $C_n B_m$. This completes the proof. □

Corollary 14. *For $m, n \geq 0$, we have*

$$\begin{aligned} \widehat{J}_{n+m} &= \widehat{J}_n \sum_{k=0}^{m+1}\sum_{l=k}^{m+1}\sum_{p=l}^{m+1} \widehat{S}_k + \widehat{J}_{n-1}\left(-13\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{S}_k + 14\sum_{k=0}^{m-1}\sum_{l=k}^{m-1}\sum_{p=l}^{m-1}\widehat{S}_k - 7\sum_{k=0}^{m-2}\sum_{l=k}^{m-2}\sum_{p=l}^{m-2}\widehat{S}_k + 3\sum_{k=0}^{m-3}\sum_{l=k}^{m-3}\sum_{p=l}^{m-3}\widehat{S}_k\right) \\ &+\widehat{J}_{n-2}\left(14\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{S}_k - 7\sum_{k=0}^{m-1}\sum_{l=k}^{m-1}\sum_{p=l}^{m-1}\widehat{S}_k + 3\sum_{k=0}^{m-2}\sum_{l=k}^{m-2}\sum_{p=l}^{m-2}\widehat{S}_k\right) \\ &+\widehat{J}_{n-3}\left(-7\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{S}_k + 3\sum_{k=0}^{m-1}\sum_{l=k}^{m-1}\sum_{p=l}^{m-1}\widehat{S}_k\right) + 3\widehat{J}_{n-4}\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{S}_k, \end{aligned}$$

$$\begin{aligned} \widehat{j}_{n+m} &= \widehat{j}_n \sum_{k=0}^{m+1}\sum_{l=k}^{m+1}\sum_{p=l}^{m+1} \widehat{S}_k \\ &+\widehat{j}_{n-1}\left(-13\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{S}_k + 14\sum_{k=0}^{m-1}\sum_{l=k}^{m-1}\sum_{p=l}^{m-1}\widehat{S}_k - 7\sum_{k=0}^{m-2}\sum_{l=k}^{m-2}\sum_{p=l}^{m-2}\widehat{S}_k + 3\sum_{k=0}^{m-3}\sum_{l=k}^{m-3}\sum_{p=l}^{m-3}\widehat{S}_k\right) \\ &+\widehat{j}_{n-2}\left(14\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{S}_k - 7\sum_{k=0}^{m-1}\sum_{l=k}^{m-1}\sum_{p=l}^{m-1}\widehat{S}_k + 3\sum_{k=0}^{m-2}\sum_{l=k}^{m-2}\sum_{p=l}^{m-2}\widehat{S}_k\right) \\ &+\widehat{j}_{n-3}\left(-7\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{S}_k + 3\sum_{k=0}^{m-1}\sum_{l=k}^{m-1}\sum_{p=l}^{m-1}\widehat{S}_k\right) + 3\widehat{j}_{n-4}\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{S}_k, \end{aligned}$$

$$
\begin{aligned}
\widehat{S}_{n+m} &= \widehat{S}_{n}\sum_{k=0}^{m+1}\sum_{l=k}^{m+1}\sum_{p=l}^{m+1}\widehat{S}_{k} \\
&+\widehat{S}_{n-1}\left(-13\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{S}_{k}+14\sum_{k=0}^{m-1}\sum_{l=k}^{m-1}\sum_{p=l}^{m-1}\widehat{S}_{k}-7\sum_{k=0}^{m-2}\sum_{l=k}^{m-2}\sum_{p=l}^{m-2}\widehat{S}_{k}+3\sum_{k=0}^{m-3}\sum_{l=k}^{m-3}\sum_{p=l}^{m-3}\widehat{S}_{k}\right) \\
&+\widehat{S}_{n-2}\left(14\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{S}_{k}-7\sum_{k=0}^{m-1}\sum_{l=k}^{m-1}\sum_{p=l}^{m-1}\widehat{S}_{k}+3\sum_{k=0}^{m-2}\sum_{l=k}^{m-2}\sum_{p=l}^{m-2}\widehat{S}_{k}\right) \\
&+\widehat{S}_{n-3}\left(-7\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{S}_{k}+3\sum_{k=0}^{m-1}\sum_{l=k}^{m-1}\sum_{p=l}^{m-1}\widehat{S}_{k}\right)+3\widehat{S}_{n-4}\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{S}_{k},
\end{aligned}
$$

and

$$
\begin{aligned}
\widehat{R}_{n+m} &= \widehat{R}_{n}\sum_{k=0}^{m+1}\sum_{l=k}^{m+1}\sum_{p=l}^{m+1}\widehat{S}_{k} \\
&+\widehat{R}_{n-1}\left(-13\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{S}_{k}+14\sum_{k=0}^{m-1}\sum_{l=k}^{m-1}\sum_{p=l}^{m-1}\widehat{S}_{k}-7\sum_{k=0}^{m-2}\sum_{l=k}^{m-2}\sum_{p=l}^{m-2}\widehat{S}_{k}+3\sum_{k=0}^{m-3}\sum_{l=k}^{m-3}\sum_{p=l}^{m-3}\widehat{S}_{k}\right) \\
&+\widehat{R}_{n-2}\left(14\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{S}_{k}-7\sum_{k=0}^{m-1}\sum_{l=k}^{m-1}\sum_{p=l}^{m-1}\widehat{S}_{k}+3\sum_{k=0}^{m-2}\sum_{l=k}^{m-2}\sum_{p=l}^{m-2}\widehat{S}_{k}\right) \\
&+\widehat{R}_{n-3}\left(-7\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{S}_{k}+3\sum_{k=0}^{m-1}\sum_{l=k}^{m-1}\sum_{p=l}^{m-1}\widehat{S}_{k}\right)+3\widehat{R}_{n-4}\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{S}_{k}.
\end{aligned}
$$

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