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Odd Lomax-Kumaraswamy Distribution: Its Properties and Applications

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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ABSTRACT

This article uses the odd Lomax-G family of distributions to study a new extension of the Kumaraswamy distribution called "odd Lomax-Kumaraswamy distribution". In this article, the density and distribution functions of the odd Lomax-Kumaraswamy distribution are defined and studied with many other properties of the distribution such as the ordinary moments, moment generating function, characteristic function, quantile function, reliability functions, order statistics and other useful measures. The model parameters are estimated by the method of maximum likelihood. The goodness-of-fit of the proposed distribution is demonstrated using two real data sets.

Keywords: Lomax-G family; Kumaraswamy distribution; odd Lomax-Kumaraswamy distribution; statistical properties; parameter estimation; goodness-of-fit.

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1. INTRODUCTION

The Kumaraswamy distribution describes a distribution in which outcomes are limited to a specific range, the probability density function within this range being characterized by two shape parameters. It is similar to the beta distribution but possibly easier to use because it has simpler analytical expressions for its probability density function and cumulative distribution function. The Kumaraswamy probability distribution was originally proposed by [1] for double bounded random processes for hydrological applications. The Kumaraswamy distribution is a family of continuous probability distributions defined on the interval [0,1] with cumulative distribution function (*cdf*) given by

$$G(x) = 1 - \left(1 - x^{\theta}\right)^{\lambda} \tag{1}$$

And the corresponding probability density function (*pdf*) given by

$$g(x) = \theta\lambda x^{\theta-1}\left(1 - x^{\theta}\right)^{\lambda-1} \tag{2}$$

For $0 \le x \le 1$, where $\theta > 0$ and $\lambda > 0$ are the shape parameters.

There are several methods of extending standard distributions and among the very recent ones are Logistic-X family [2], a new Weibull-G family [3], a Lindley-G family [4], a Gompertz-G family [5], an odd Lindley-G family [6] and an odd Lomax generator of distributions (Odd Lomax-G family) [7].

Due to this families and many others, numerous compound models have been introduced for modeling statistical data and these compound distributions are found to be skewed, flexible and more better in statistical modeling than the standard counterparts [8-22].

These families of distribution have also led to the development of some generalizations of the Kumaraswamy distribution in the literature such as the transmuted Kumaraswamy distribution [23], the exponentiated Kumaraswamy distribution [24], the Kumaraswamy-Kumaraswamy distribution [25] and the Lomax-Kumaraswamy distribution [26].

The following is the arrangement of remaining sections in this article: The new model with graphical representation is given in section 2. Section 3 derived some properties of the new distribution. The distribution of order statistics is presented in section 4. The estimation of unknown parameters of the distribution using maximum likelihood estimation is provided in section 5. An application of the new model to two real life data sets is done in section 6 and a summary and conclusion of the study is given in section 7.

2. ODD LOMAX-KUMARASWAMY DISTRIBUTION (OLomKumD)

According to [7], the cumulative distribution function (cdf) and the probability density function (pdf) of the odd Lomax generator (Lomax-G family) of distributions with two extra shape parameters ($\alpha > 0$ and $\beta > 0$) are defined by:

$$F(x) = \alpha\beta^{\alpha}\int_{0}^{\frac{G(x)}{1-G(x)}}(\beta+t)^{-\alpha-1}dt = 1-\beta^{\alpha}\left\{\beta+\frac{G(x)}{1-G(x)}\right\}^{-\alpha} \tag{3}$$

and

$$f(x) = \alpha\beta^{\alpha}\frac{g(x)}{\left[1-G(x)\right]^{2}}\left\{\beta+\frac{G(x)}{1-G(x)}\right\}^{-(\alpha+1)} \tag{4}$$

respectively, where $g(x)$ and $G(x)$ represent the pdf and the cdf of the continuous distribution to be modified respectively.

Substituting equation (1) and (2) in (3) and (4) above and simplifying, we obtain the cdf and pdf of the OLomKumD for a random variable *X* as:

$$F(x) = 1-\beta^{\alpha}\left[\beta+\left(1-x^{\theta}\right)^{-\lambda}-1\right]^{-\alpha} \tag{5}$$

And

$$f(x) = \alpha\beta^{\alpha}\theta\lambda x^{\theta-1}\left(1-x^{\theta}\right)^{-\lambda-1}\left[\beta+\left(1-x^{\theta}\right)^{-\lambda}-1\right]^{-\alpha-1} \tag{6}$$

"respectively, for $\alpha, \beta, \theta, \lambda > 0$ and $0 \le x \le 1$.

The plot of the pdf of the OLomKumD using some parameter values are displayed in Fig. 1 as follows.

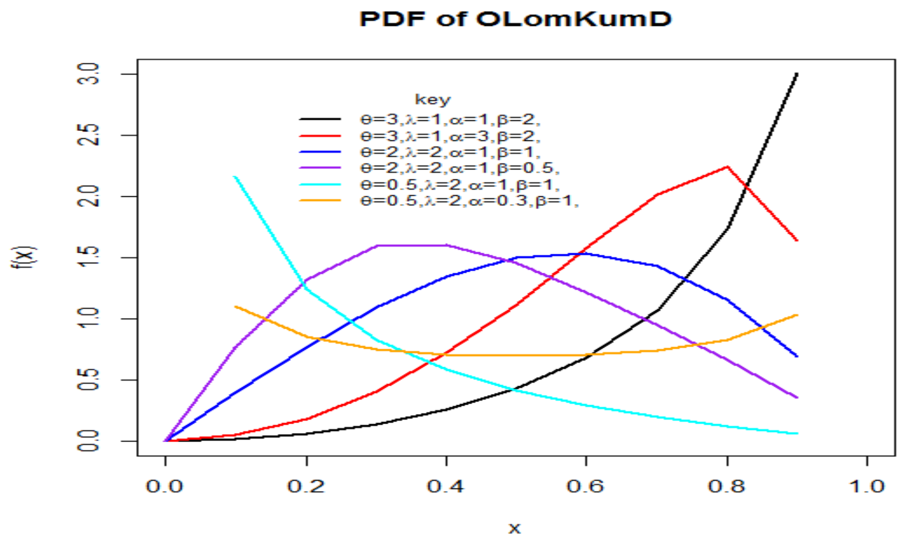


Fig. 1. PDF of OLomKumD

3. PROPERTIES OF ODD LOMAX-KUMARASWAMY DISTRIBUTION

This section contains derivations and discussions of some properties of the proposed distribution. These are presented as follows:

3.1 Moments

Let X denote a continuous random variable, the n^{th} moment of X is given by;

$$\mu_n^{'} = E\left(X^n\right) = \int_0^\infty x^n f(x)dx \tag{7}$$

where f(x), the *pdf* of the OLomKumD is as given in equation (6) as:

$$f\left(x\right) = \alpha\beta\theta\lambda x^{\theta-1}\left(1-x^{\theta}\right)^{-\lambda-1}\left[\beta+\left(1-x^{\theta}\right)^{-\lambda}-1\right]^{-\alpha-1}$$

Simplifying the pdf above results in the following:

$$f\left(x\right) = \alpha\beta^{-1}\theta\lambda x^{\theta-1}\left(1-x^{\theta}\right)^{-\lambda-1}\left[1+\beta^{-1}\left(1-x^{\theta}\right)^{-\lambda}\left(1-\left(1-x^{\theta}\right)^{\lambda}\right)\right]^{-\alpha-1} \tag{8}$$

Before substituting (8) in (7), we perform the expansion and simplification and linear representation of the pdf as follows:

First, by using power series expansion on the last term in (8), we obtain:

$$\left[1+\beta^{-1}\left(1-x^{\theta}\right)^{-\lambda}\left(1-\left(1-x^{\theta}\right)^{\lambda}\right)\right]^{-\alpha-1} = \sum_{k=0}^{\infty}\binom{-\alpha-1}{k}\beta^{-k}\left(1-x^{\theta}\right)^{-\lambda k}\left[1-\left(1-x^{\theta}\right)^{\lambda}\right]^{k} \tag{9}$$

Making use of the result in (9) above and simplifying, equation (8) becomes

$$f(x)=\sum_{k=0}^{\infty}\binom{-\alpha-1}{k}\frac{\alpha\theta\lambda}{\beta^{k+1}}x^{\theta-1}\left(1-x^{\theta}\right)^{-\lambda(k+1)-1}\left[1-\left(1-x^{\theta}\right)^{\lambda}\right]^{k} \tag{10}$$

Also using generalized binomial expansion gives:

$$\left[1-\left(1-x^{\theta}\right)^{\lambda}\right]^{k}=\sum_{m=0}^{\infty}(-1)^{m}\binom{k}{m}\left(1-x^{\theta}\right)^{\lambda m} \tag{11}$$

Making use of the result in (11) above in equation (10) and simplifying, we obtain:

$$f(x)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\binom{k}{m}\binom{-\alpha-1}{k}\frac{\alpha\theta\lambda}{(-1)^{-m}\beta^{k+1}}x^{\theta-1}\left(1-x^{\theta}\right)^{\lambda(m-k-1)-1} \tag{12}$$

Now, let $\eta_{km}=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\binom{k}{m}\binom{-\alpha-1}{k}\frac{\alpha(-1)^{m}}{\beta^{k+1}}$ be a constant, which implies that the pdf in (12) can also be written in its simple and linear form as:

$$f(x)=\eta_{km}\theta\lambda x^{\theta-1}\left(1-x^{\theta}\right)^{\lambda(m-k-1)-1} \tag{13}$$

Now, using the linear representation of the pdf of the OLomKumD in equation (13), the n^th ordinary moment of the OLomKumD is represented as:

$$\mu_n^{'}=E\left(X^{n}\right)=\int_{0}^{1}x^{n}f(x)\,dx=\int_{0}^{1}\eta_{km}\theta\lambda x^{n+\theta-1}\left(1-x^{\theta}\right)^{\lambda(m-k-1)-1}dx \tag{14}$$

Recall that for the Kumaraswamy distribution, the r^th ordinary moment is given as:

$$\mu_r^{'}=E\left(X^{r}\right)=\int_{0}^{1}x^{r}f(x)dx=\theta\lambda\int_{0}^{1}x^{r+\theta-1}\left(1-x^{\theta}\right)^{\lambda-1}dx=\lambda B\left(\frac{r}{\theta}+1,\lambda\right) \tag{15}$$

Therefore, the n^th ordinary moment of the OLomKumD can be expressed from (15) as:

$$\mu_n^{'}=E\left(X^{n}\right)=\int_{0}^{1}\eta_{km}\theta\lambda x^{n+\theta-1}\left(1-x^{\theta}\right)^{\lambda(m-k-1)-1}dx=\lambda B\left(\frac{n}{\theta}+1,\lambda(m-k-1)\right)$$

$$\mu_n^{'}=E\left(X^{n}\right)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\binom{k}{m}\binom{-\alpha-1}{k}\frac{\alpha(-1)^{m}}{\beta^{k+1}}\lambda B\left(\frac{n}{\theta}+1,\lambda(m-k-1)\right) \tag{16}$$

The mean ($\mu_1^{'}$), variance (σ^2), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) can be calculated from the ordinary and non-central moments using some well-known relationships such as:

$$\mu_1^{'} = E\left(X\right), Var(X) = \sigma^2 = \mu_2^{'} - \left\{\mu_1^{'}\right\}^2,\ CV = \left\{\frac{\sigma^2}{\left(\mu_1^{'}\right)^2}\right\}^{\frac{1}{2}} \quad CS = E\left(\frac{x-\mu_1^{'}}{\sigma}\right)^3 = \frac{\mu_3}{(\sigma)^3}$$

and

$$CK = E\left(\frac{x-\mu_1^{'}}{\sigma}\right)^4 = \frac{\mu_4}{(\sigma)^4}\ .$$

3.2 Moment Generating Function

The moment generating function of a random variable X can be obtained as

$$M_x(t) = E\left[e^{tx}\right] = \int_{-\infty}^{\infty} e^{tx} f(x)dx \tag{17}$$

Recall that by power series expansion,

$$e^{tx} = \sum_{r=0}^{\infty}\frac{(tx)^r}{r!} = \sum_{r=0}^{\infty}\frac{t^r}{r!}x^r \tag{18}$$

Therefore, the moment generating function can also be expressed as:

$$M_x\left(t\right) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^1 x^r f\left(x\right)dx = \sum_{r=0}^{\infty}\frac{t^r}{r!}E\left(X^r\right) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\left[\mu_r^{'}\right]$$

Using the result in equation (18) and simplifying the integral in (17) therefore we have:

$$M_x\left(t\right) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\left[\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\binom{k}{m}\binom{-\alpha-1}{k}\frac{\alpha\left(-1\right)^m}{\beta^{k+1}}\lambda B\left(\frac{r}{\theta}+1, \lambda\left(m-k-1\right)\right)\right] \tag{19}$$

3.3 Characteristic Function

A representation for the characteristic function is given by

$$\phi_x\left(t\right) = E\left(e^{itx}\right) = \int_0^1 e^{itx} f\left(x\right)dx \tag{20}$$

Recall that by power series expansion,

$$e^{itx} = \sum_{r=0}^{\infty}\frac{\left(itx\right)^r}{r!} = \sum_{r=0}^{\infty}\frac{\left(it\right)^r}{r!}x^r \tag{21}$$

Hence, simple algebra and use of (21) above produces the following results:

$$\phi_x(t)=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\int_0^1 x^r f(x)\,dx=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}E\left(X^r\right)=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\left[\mu_r^{'}\right]$$

$$\phi_x(t)=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\left[\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\binom{k}{m}\binom{-\alpha-1}{k}\frac{\alpha(-1)^m}{\beta^{k+1}}\lambda B\left(\frac{r}{\theta}+1,\lambda(m-k-1)\right)\right] \tag{22}$$

3.4 Quantile Function, Median and Simulation

According to [27], the quantile function for any distribution is defined in the form $Q(u)=F^{-1}(u)$ where $Q(u)$ is the quantile function of *F*(*x*) for $0<u<1$

To derive the quantile function of the OLomKumD, the cdf of the OLomKumD is considered and inverted according to the above definition as follows:

$$F(x)=1-\beta^{\alpha}\left[\beta+\left(1-x^{\theta}\right)^{-\lambda}-1\right]^{-\alpha}=u \tag{23}$$

Simplifying equation (23) above gives:

$$\left(1-x^{\theta}\right)^{-\lambda}=1-\beta+\beta(1-u)^{-\frac{1}{\alpha}} \tag{24}$$

Collecting like terms in equation (24) and simplifying the result, the quantile function of the OLomKumD is obtained as:

$$Q(u)=\left\{1-\left[1-\beta+\beta(1-u)^{-\frac{1}{\alpha}}\right]^{-\frac{1}{\lambda}}\right\}^{\frac{1}{\theta}} \tag{25}$$

where *u* is a uniform variate on the unit interval (0,1).

Using (25) above, the median of *X* from the OLomKumD is simply obtained by setting *u*=0.5 and this substitution of $u=0.5$ in Equation (25) gives:

$$MD=\left\{1-\left[1-\beta+\beta(0.5)^{-\frac{1}{\alpha}}\right]^{-\frac{1}{\lambda}}\right\}^{\frac{1}{\theta}} \tag{26}$$

Similarly, random numbers can be simulated from the OLomKumD by setting $Q(u)=X$ and this process is called simulation using inverse transformation method. This means for any values of the parameters $\alpha,\beta,\theta,\lambda>0$ and $u\in(0,1)$:

$$X=\left\{1-\left[1-\beta+\beta(1-u)^{-\frac{1}{\alpha}}\right]^{-\frac{1}{\lambda}}\right\}^{\frac{1}{\theta}} \tag{27}$$

“where u is a uniform variate on the unit interval (0,1).

Again from the above quantile function, the quantile based measures of skewness and kurtosis are obtained as follows:

[28] defined the Bowley's measure of skewness based on quartiles as:

$$SK = \frac{Q\left(3/4\right) - 2Q\left(1/2\right) + Q\left(1/4\right)}{Q\left(3/4\right) - Q\left(1/4\right)} \tag{28}$$

And [29] presented the Moors' kurtosis based on octiles as:

$$KT = \frac{Q\left(7/8\right) - Q\left(5/8\right) - Q\left(3/8\right) + \left(1/8\right)}{Q\left(6/8\right) - Q\left(1/8\right)} \tag{29}$$

"where $Q(.)$ is calculated by using the quantile function from equation (25).

3.5 Reliability Analysis of the OLomKumD

In this section, the survival (or reliability) function, the hazard (or failure) rate function, the cumulative hazard function, the reverse hazard function and the odds function are obtained for the OLomKumD as follows:

The Survival function describes the probability that a unit, component or an individual will not fail after a given time. Mathematically, the survival function is given by:

$$S(x) = 1 - F(x) \tag{30}$$

Using the *cdf* of the OLomKumD in (30) and simplifying the result, the survival function for the OLKD is obtained as:

$$S(x) = \beta^{\alpha}\left[\beta + \left(1 - x^{\theta}\right)^{-\lambda} - 1\right]^{-\alpha} \tag{31}$$

The figure below presents some plots for the survival function of the OLomKumD using different parameter values.

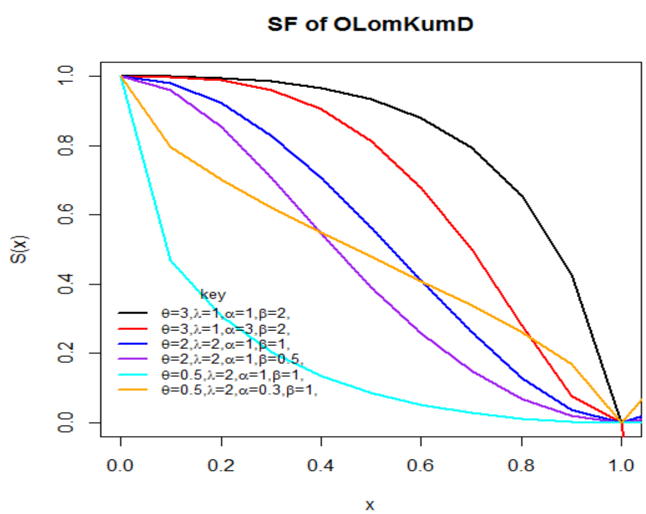


Fig. 2. Survival function of OLomKumD

Hazard function is also called failure rate function and it represents the likelihood that a component will fail for an interval of time. The hazard function is defined as:

$$h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)} \tag{32}$$

Making use of the pdf and cdf of OLomKumD, an expression for the hazard rate of the OLomKumD is simplified and given by:

$$h(x)=\frac{\alpha\beta^{\alpha}\theta\lambda x^{\theta-1}\left(1-x^{\theta}\right)^{-\lambda-1}\left[\beta+\left(1-x^{\theta}\right)^{-\lambda}-1\right]^{-\alpha-1}}{\beta^{\alpha}\left[\beta+\left(1-x^{\theta}\right)^{-\lambda}-1\right]^{-\alpha}}$$

$$h(x)=\alpha\theta\lambda x^{\theta-1}\left(1-x^{\theta}\right)^{-\lambda-1}\left[\beta+\left(1-x^{\theta}\right)^{-\lambda}-1\right]^{-1} \tag{33}$$

where $x>0,\alpha,\beta,\theta>0$.

The following figure displays useful plots of the hazard function of OLomKumD for some arbitrary parameter values.

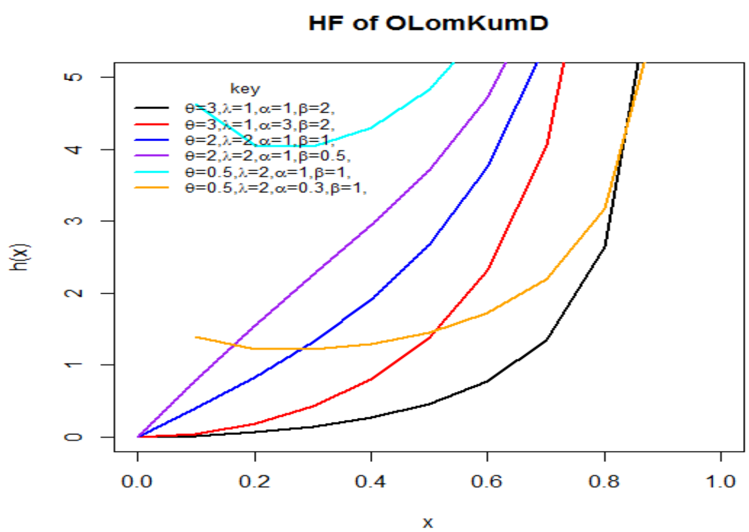


Fig. 3. The hazard function of OLomKumD

The cumulative hazard function of a variable or unit is a function that generates a cumulative hazard value which corresponds to the sum of all the hazard values for failed units with ranks up to and including that failed unit. The cumulative hazard function is defined as:

$$H(x)=\int_{0}^{x}h(x)dt=\int_{0}^{x}\frac{f(t)}{1-F(t)}dt=-\ln S(x) \tag{34}$$

Substituting the cdf of the OLomKumD in (34), the cumulative hazard function for the OLomKumD is obtained as:

$$H(x)=-\ln\left\{\beta^{\alpha}\left[\beta+\left(1-x^{\theta}\right)^{-\lambda}-1\right]^{-\alpha}\right\} \tag{35}$$

The reversed hazard rate (Rh(x)) is defined as the ratio of the density function to the distribution function of a random variable. The reversed hazard function of a variable is mathematically defined as:

$$Rh(x)=\frac{f(x)}{F(x)} \tag{36}$$

Again, substituting the pdf and cdf of the OLomKumD in (36) and simplifying, the reverse hazard function of the OLomKumD is expressed as:

$$Rh(x)=\frac{\alpha\beta\theta\lambda x^{\theta-1}\left(1-x^{\theta}\right)^{-\lambda-1}\left[\beta+\left(1-x^{\theta}\right)^{-\lambda}-1\right]^{-\alpha-1}}{1-\beta^{\alpha}\left[\beta+\left(1-x^{\theta}\right)^{-\lambda}-1\right]^{-\alpha}} \tag{37}$$

The odds function of a random variable X is a measure of the ratio of the probability that the variable or unit will survive beyond x to the probability that it will fail before x. It is obtained by dividing the cdf by the reliability (survival) function. That is:

$$O(x)=\frac{F(x)}{1-F(x)}=\frac{F(x)}{S(x)} \tag{38}$$

Using the cdf of the OLomKumD in (38), the odds function for the distribution is given as:

$$O(x)=\beta^{-\alpha}\left[\beta+\left(1-x^{\theta}\right)^{-\lambda}-1\right]^{\alpha}-1 \tag{39}$$

where $\alpha,\beta,\theta,\lambda>0$ and $0\le x\le 1$.

4. ORDER STATISTICS

Suppose $X_1,X_2,.....,X_n$ is a random sample from the OLomKumD and let $X_{1:n},X_{2:n},.....,X_{i:n}$ denote the corresponding order statistic obtained from this same sample. The *pdf*, $f_{a:n}(x)$ of the a^{th} order statistic can be obtained by:

$$f_{a:n}(x)=\frac{n!}{(a-1)!(n-a)!}\sum_{k=0}^{n-a}(-1)^{k}\binom{n-a}{k}f(x)F(x)^{k+a-1} \tag{40}$$

Using (5) and (6), the pdf of the a^{th} order statistics $X_{a:n}$, can be expressed from (40) as:

$$f_{a:n}(x)=\frac{n!}{(a-1)!(n-a)!}\sum_{k=0}^{n-a}(-1)^{k}\binom{n-a}{k}\left[\frac{\alpha\beta^{\alpha}\theta\lambda x^{\theta-1}\left(1-x^{\theta}\right)^{-\lambda-1}}{\left[\beta+\left(1-x^{\theta}\right)^{-\lambda}-1\right]^{\alpha+1}}\right]\left[1-\beta^{\alpha}\left[\beta+\left(1-x^{\theta}\right)^{-\lambda}-1\right]^{-\alpha}\right]^{k+a-1} \tag{41}$$

Hence, the *pdf* of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the OLomKumD are respectively given by:

$$f_{1:n}(x)=\frac{n!}{(n-1)!}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\left[\frac{\alpha\beta^{\alpha}\theta\lambda x^{\theta-1}\left(1-x^{\theta}\right)^{-\lambda-1}}{\left[\beta+\left(1-x^{\theta}\right)^{-\lambda}-1\right]^{\alpha+1}}\right]\left[1-\beta^{\alpha}\left[\beta+\left(1-x^{\theta}\right)^{-\lambda}-1\right]^{-\alpha}\right]^{k} \tag{42}$$

and

$$f_{n:n}(x)=n\left[\frac{\alpha\beta^{\alpha}\theta\lambda x^{\theta-1}\left(1-x^{\theta}\right)^{-\lambda-1}}{\left[\beta+\left(1-x^{\theta}\right)^{-\lambda}-1\right]^{\alpha+1}}\right]\left[1-\beta^{\alpha}\left[\beta+\left(1-x^{\theta}\right)^{-\lambda}-1\right]^{-\alpha}\right]^{n-1} \tag{43}$$

5. MAXIMUM LIKELIHOOD ESTIMATION (MLE) OF THE UNKNOWN PARAMETERS OF THE OLomKumD

Let $X_1, X_2, \ldots\ldots\ldots, X_n$ be a sample of size 'n' independently and identically distributed random variables from the OLomKumD with unknown parameters, α, β, θ and λ defined previously.

The likelihood function of the OLomKumD using the pdf is given by:

$$L(X\mid\alpha,\beta,\theta,\lambda)=\left(\alpha\beta^{\alpha}\theta\lambda\right)^{n}\prod_{i=1}^{n}\left\{x_i^{\theta-1}\left(1-x_i^{\theta}\right)^{-\lambda-1}\left[\beta+\left(1-x_i^{\theta}\right)^{-\lambda}-1\right]^{-\alpha-1}\right\} \tag{44}$$

Let the natural logarithm of the likelihood function be, $l=\log L(X\mid\alpha,\beta,\theta,\lambda)$, therefore, taking the natural logarithm of the function equation (44) above gives:

$$l=n\log\alpha+n\alpha\log\beta+n\log\theta+n\log\lambda+(\theta-1)\sum_{i=1}^{n}\log x_i-(\lambda+1)\sum_{i=1}^{n}\log\left(1-x_i^{\theta}\right)-(\alpha+1)\sum_{i=1}^{n}\log\left[\beta+\left(1-x_i^{\theta}\right)^{-\lambda}-1\right] \tag{45}$$

Differentiating l partially with respect to α, β, θ and λ respectively gives the following results:

$$\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha}+n\log\beta-\sum_{i=1}^{n}\log\left[\beta+\left(1-x_i^{\theta}\right)^{-\lambda}-1\right] \tag{46}$$

$$\frac{\partial l}{\partial\beta}=\frac{n\alpha}{\beta}-(\alpha+1)\sum_{i=1}^{n}\left[\beta+\left(1-x_i^{\theta}\right)^{-\lambda}-1\right]^{-1} \tag{47}$$

$$\frac{\partial l}{\partial\theta}=\frac{n}{\theta}+\sum_{i=1}^{n}\log x_i+(\lambda+1)\sum_{i=1}^{n}\left\{\frac{x^{\theta}\ln x}{\left(1-x_i^{\theta}\right)}\right\}-\lambda(\alpha+1)\sum_{i=1}^{n}\left\{\frac{\left(1-x_i^{\theta}\right)^{-\lambda-1}x^{\theta}\ln x}{\left[\beta+\left(1-x_i^{\theta}\right)^{-\lambda}-1\right]}\right\} \tag{48}$$

$$\frac{\partial l}{\partial\lambda}=\frac{n}{\lambda}+\sum_{i=1}^{n}\log\left(1-x_i^{\theta}\right)+(\alpha+1)\sum_{i=1}^{n}\left\{\frac{\left(1-x_i^{\theta}\right)^{-\lambda}\ln\left(1-x_i^{\theta}\right)}{\left[\beta+\left(1-x_i^{\theta}\right)^{-\lambda}-1\right]}\right\} \tag{49}$$

Making equation (46), (47), (48) and (49) equal to zero (0) and solving for the solution of the non-linear system of equations produce the maximum likelihood estimates of parameters α, β, θ and λ. Note that it is difficult to solve the above equations analytically and therefore the Newton-

Raphson's iteration method is applied using computer packages such as Maple or R or other software.

6. APPLICATIONS

In this section, two applications to real life data are provided to illustrate the flexibility of the OLKD (odd Lomax-Kumaraswamy distribution, OLomKumD) distribution introduced in Section 2 compared to the Lomax-Kumaraswamy distribution (LomKumD), Transmuted Kumaraswamy distribution (TransKumD), Kumaraswamy-Kumaraswamy distribution (KumKumD and the conventional Kumaraswamy distribution (KumD). The maximum likelihood estimates (MLEs) of the model parameters are determined and some goodness-of-fit statistics for this distribution are compared with other competitive models.

The model selection is carried out based upon the value of the log-likelihood function evaluated at the MLEs (ℓ), Akaike Information Criterion, *AIC,* Consistent Akaike Information Criterion, *CAIC*, Bayesian Information Criterion, *BIC*, Hannan Quin Information Criterion, *HQIC*, Anderson-Darling (A*), Cramèr-Von Mises (W*) and Kolmogorov-smirnov (K-S) statistics. The details about the statistics A*, W* and K-S are discussed in [30]. Meanwhile, the smaller these statistics are, the better the fit of the distribution is. The required computations are carried out using the R package "AdequacyModel" which is freely available from http://cran.r-project.org/web/packages/AdequacyModel/AdequacyModel.pdf.

Data set I: This data is flood data with 20 observations obtained from [31] and it has been used by [23]. The summary of dataset I is also provided in Table 1. as follows:

Table 1. Summary statistics for the dataset I

n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
20	0.265	0.3345	0.4070	0.4578	0.4232	0.7400	0.0157	1.0677	0.5999

Table 2. Maximum likelihood parameter estimates for dataset I

Distribution	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$
OLomKumD	6.83963927	4.51446180	1.29483209	0.01381898
LomKumD	6.83963927	4.51446180	1.29483209	0.01381898
TransKumD	2.4464420	9.6294406	-0.9917593	
KumKumD	1.4886322	8.0584351	8.6670875	0.9112175
KumD	3.034985	9.140995		

Table 3. The statistics ℓ, AIC, CAIC, BIC and HQIC for dataset I

Distribution	$\hat{\ell}$	*AIC*	*CAIC*	*BIC*	*HQIC*	Ranks
OLomKumD	-15.00556	-22.01111	-19.34444	-18.02818	-21.2336	2nd
LomKumD	-15.09483	-22.18967	-19.523	-18.20674	-21.41216	1st
TransKumD	-13.80434	-21.60868	-20.10868	-18.62149	-21.02555	3rd
KumKumD	-14.73712	-21.47423	-18.80757	-17.4913	-20.69672	5th
KumD	-12.70427	-21.40853	-20.70265	-19.41707	-21.01978	4th

Table 4. The A^*, W^*, K-S statistic and P-values for dataset I

Distribution	A^*	W^*	*K-S*	*P-Value (K-S)*	Ranks
OLomKumD	0.3891962	0.05917019	0.13095	0.8827	2nd
LomKumD	0.364724	0.05454672	0.13262	0.8733	1st
TransKumD	0.7982341	0.1337387	0.18869	0.4747	3rd
KumKumD	0.529415	0.08648492	0.21196	0.3301	5th
KumD	0.9272041	0.1575337	0.19794	0.4134	4th

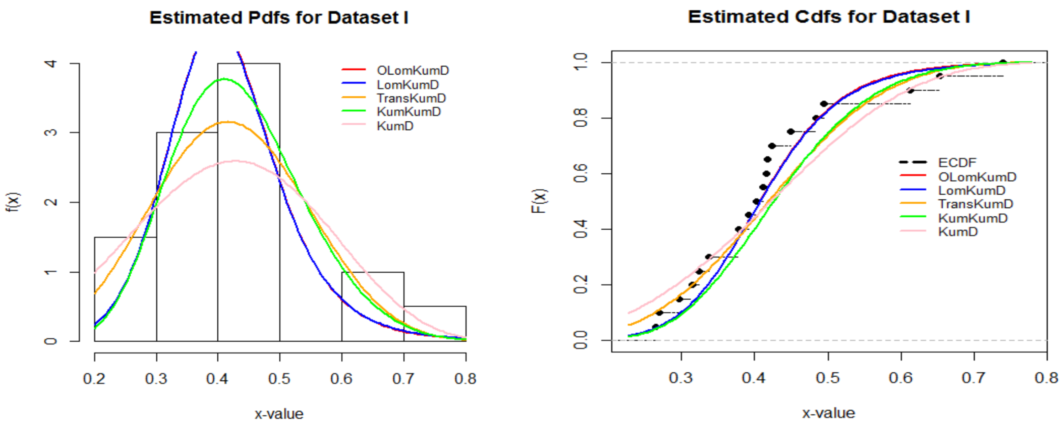


Fig. 4. Histogram and plots of the estimated densities and cdfs of the OLomKumD and other fitted distributions to dataset I

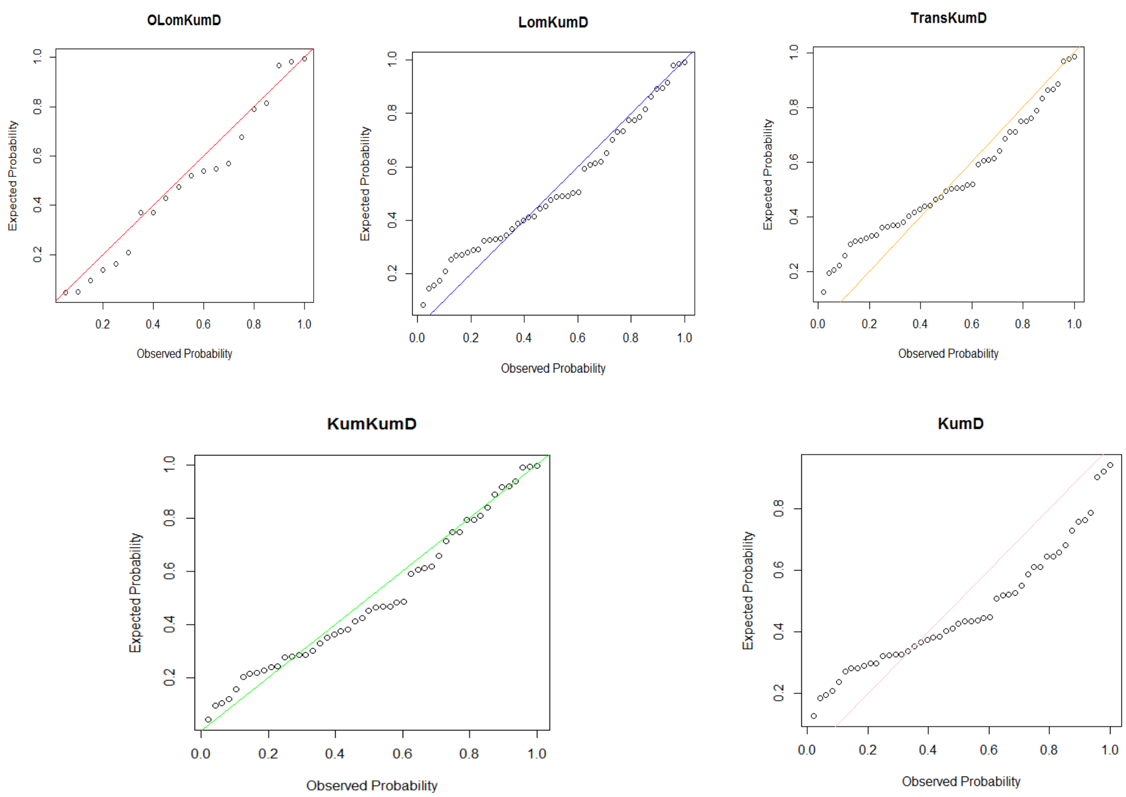


Fig. 5. Probability plots for the fit of the OLomKumD and other fitted models based on dataset I

Table 5. Summary statistics for dataset II

n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
48	0.0903	0.1623	0.1988	0.2627	0.2181	0.4641	0.0069	1.1694	1.1099

Table 6. Maximum likelihood parameter estimates for dataset II

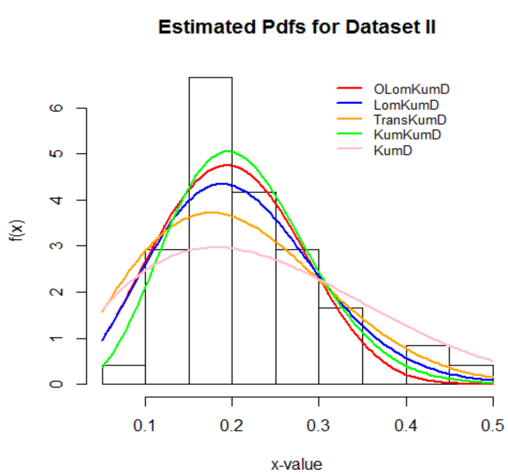
Distribution	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$
OLomKumD	2.605155	9.966554	8.054591	1.751369
LomKumD	2.605155	9.966554	8.054591	1.751369
TransKumD	2.064456	9.658148	0.976459	
KumKumD	1.220560	5.931575	3.360720	3.327903
KumD	1.773324	9.608201		

Table 7. The statistics ℓ, AIC, CAIC, BIC and HQIC for dataset II

Distribution	$\hat{\ell}$	***AIC***	***CAIC***	***BIC***	***HQIC***	**Ranks**
OLomKumD	-49.35205	-90.70411	-89.77387	-83.2193	-87.87559	4th
LomKumD	-52.91954	-97.83908	-96.90885	-90.35428	-95.01056	2nd
TransKumD	-49.74191	-93.48382	-92.93836	-87.87021	-91.36243	3rd
KumKumD	-54.66335	-101.3267	-100.3965	-93.8419	-98.49818	1st
KumD	-44.85103	-85.70206	-85.43539	-81.95966	-84.2878	5th

Table 8. The A^*, W^*, K-S statistic and P-values for dataset II

Distribution	***A****	***W****	***K-S***	**P-Value (K-S)**	**Ranks**
OLomKumD	1.571814	0.2548545	0.15772	0.1835	4th
LomKumD	0.9413881	0.1556414	0.1506	0.2264	2nd
TransKumD	1.015258	0.1662217	0.19559	0.05082	3rd
KumKumD	0.11685	0.9102744	0.11685	0.5287	1st
KumD	0.9136833	0.1497096	0.17698	0.09887	5th



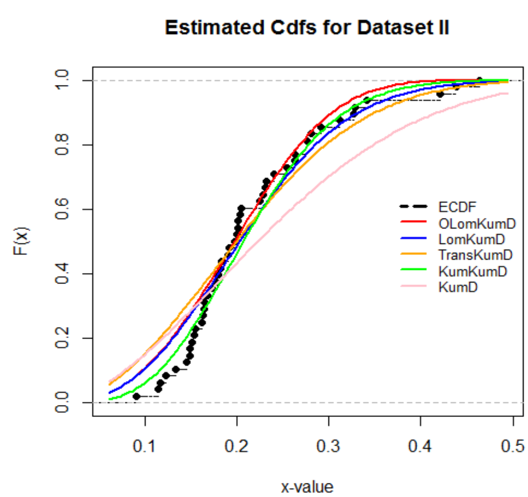


Fig. 6. Histogram and plots of the estimated densities and cdfs of the OLomKumD and other fitted distributions to dataset II

Data set II: The second data set is on shape measurements of 48 rock samples from a petroleum reservoir. This data was extracted from BP research, image analysis by Ronit Katz, u Oxford and has been used for analysis [24]. The summary of dataset II is also provided in Table 2 as follows;

Tables 2 and 6 list the values of the MLEs of the model parameters for both datasets, whereas the values of AIC, CAIC, BIC and HQIC are listed in Tables 3 and 7 for datasets I and II respectively. Also, the values of A*, W* and K-S for datasets I and II are provided in Tables 4 and 8 respectively.

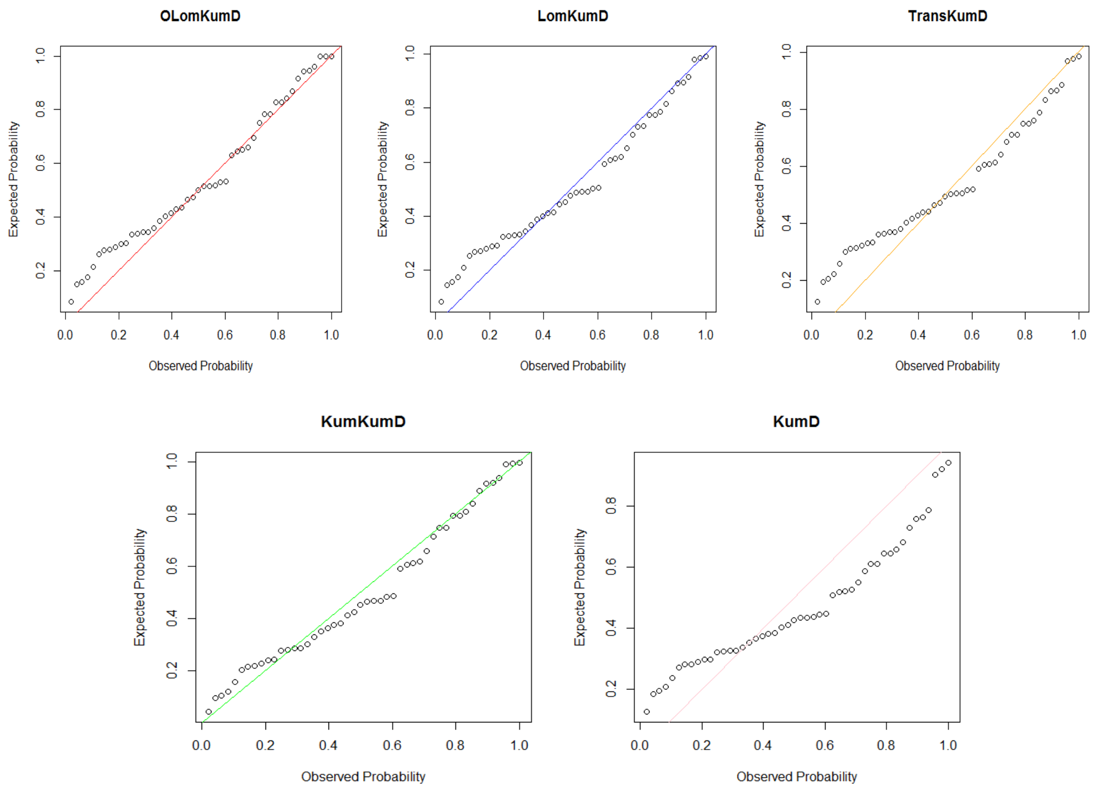


Fig. 7. Probability plots for the fit of the OLomKumD and other competing models based on dataset II

The plots of the fitted OLomKumD density and cumulative distribution with those of competing distributions for datasets I and II are displayed in Figures 5 and 7 respectively. The PP-plots of the fitted distributions are also given in Figures 6 and 7 for datasets I and II respectively. From the results in all the measures above, it was observed that there is no much difference between the Lomax-Kumaraswamy and the odd Lomax-Kumaraswamy distributions despite the slight variation in the two families. The results also show that the two distributions are better compared the three other fitted distributions (transmuted Kumaraswamy, Kumaraswamy-Kumaraswamy and the conventional Kumaraswamy distributions). Also, there is no big difference between the results from dataset I and Dataset II and therefore it is proven that the proposed distribution (odd Lomax-Kumaraswamy) is a more flexible distribution than the other existing distributions. These results are clearly confirmed by the estimated density plots and also the probability plots of the fitted distributions as shown in the figures above.

7. SUMMARY AND CONCLUSION

In this study, a four-parameter extension of the Kumaraswamy distribution is proposed called "odd Lomax-Kumaraswamy distribution". This study has looked at many important properties of the new distribution including explicit expressions for the moments, generating function, characteristics function, quantile function and related measures, reliability functions and order statistics. The maximum likelihood method has been used to estimate the parameters of the proposed distribution. The graphs of the pdf of the distribution revealed that it is flexible and that its shape various as the values of the parameters are changed. Also, the plots of the survival and hazard functions of the proposed model indicate that it will be useful for real events where probability of survival decreases with increase in age or time while that of failure increases with time. The proposed distribution with other existing distributions is fitted to two real life datasets to prove their flexibility compared to existing models. Based on some standard model selection criteria, it was found that the proposed distribution provides good fit to these datasets. The results obtained show that the odd Lomax-Kumaraswamy distribution consistently exhibits good performance than some of the other competing models. Hence, it is recommended for applications in data science and other areas of statistical research.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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