Journal of Advances in Mathematics and Computer Science

Volume 38, Issue 5, Page 60-73, 2023; Article no.JAMCS.97080

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

Equivalents of Various Principles of Zermelo, Zorn, Ekeland, Caristi and Others

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/JAMCS/2023/v38i51761

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/97080

Received: 03/01/2023
Accepted: 28/02/2023
Published: 04/03/2023

Original Research Article

Abstract

Motivated by the Ekeland variational principle, we obtained a Metatheorem in 1985-87 stating that some well-known existence of maximal elements can be equivalently formulated to existence theorems on fixed elements, stationary points, common fixed points, common stationary points, and others. In the present article, we introduce our new 2023 Metatheorem and its applications to various theorems due to Zermelo, Zorn, Ekeland, Caristi, and related results. In fact, this is a historical supplement of our previous article entitled "Foundations of Ordered Fixed Point Theory."

Keywords: Preorder; metric space; fixed point; stationary point; maximal element.

Mathematics Subject Classification (2020): 03E04, 03E25, 06A06, 06A75, 47H10, 54E35, 54H25, 58E30, 65K10.

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1 Introduction

Since the appearances of the Ekeland variational principle [1-3] in 1972-79 and the Caristi fixed point theorem [4] in 1976, nearly one thousand works followed on their equivalents, generalizations, modifications, applications, and related topics. Many of them are concerned with new spaces extending complete metric spaces, new metrics or topologies on them, and new order relations extending the so-called Caristi order.

While the author was working on the Ekeland principle and the Caristi theorem in 1984-2000, in order to give some equivalents of them, we obtained a Metatheorem [5-10] in 1985-2000 on fixed point theorems related to the order theory. It claims that certain order theoretic maximal element statements are equivalent to theorems on fixed points, stationary points, common fixed points, common stationary points of families of maps or multimaps. As usual in the mathematical community, our Metatheorem was attracted a little for a long period.

Later in 2022, we came back to our Metatheorem after 22 years have passed and obtained its extended versions in [11-15] with a large number of their consequences [11-20]. These are applied to the traditional order theoretic results and, consequently, there have appeared the so-called Ordered Fixed Point Theory [15]. This can be comparable to traditional several fields in the fixed point theory, that is, Analytical fixed point theory is originated from Brouwer in 1912 and concerns mainly with topological vector spaces; Metric fixed point theory is originated from Banach in 1922 and deals with generalizations of contractions and nonexpansive maps; and Topological fixed point theory relates mainly with original works of Lefschetz, Nielsen, and Reidemeister.

In our previous work entitled "Foundations of Ordered Fixed Point Theory"[15] in 2022, we established a large number of improved versions of historically well-known maximal element theorems and fixed point theorems related to order structure. It is based on our 2023 Metatheorem and the Brøndsted-Jachymski Principle established by ourselves in 2022.

In the present article, we introduce our new 2023 Metatheorem in [21,22], its short history, its applications to various theorems of Zermelo, Zorn, Ekeland, Caristi, Takahashi, and related results. In fact, this is a historical supplement of [15] and organized as follows.

Section 2 is to introduce the Brøndsted-Jachymski Principle and its applications to improve the Zermelo fixed point theorem. In Section 3, we introduce improved versions of Zorn's Lemma and an example. Section 4 devotes to introduce an improved Caristi fixed point theorem due to Chen-Cho-Yang [23] and its elementary proof based on our new theorem in Section 2. Section 5 is concerned with the dual forms of the Caristi theorem based on Lin-Du [24].

In Section 6, we derive Maximal (resp. Minimal) Element Principle in [18,19] from our new 2023 Metatheorem and a similar theorem from Metatheorem* in [21]. We also improve Jachymski's 2003 Theorem [25] on converses to theorems of Zsrmelo and Caristi. Section 7 devotes equivalent formulations and extensions of the weak Ekeland Principle, Caristi-Kirk's Theorem, and Takahashi's minimization principle. In Section 8, we analyze how to apply general theorems in Section 6 to results of (1) Zermelo type, (2) Zorn type, (3) Caristi type, (4) dual Caristi type, and (5) Ekeland type in this article.

Finally, Section 8 devotes to epilogue.

2 Extended Zermelo Fixed Point Theorem

A preorder is reflexive and transitive. A partial order is reflexive, antisymmetric, and transitive. A chain or a simply ordered set is a partially ordered set with an extra condition that any two elements are comparable. A well order is a simple order such that every subset has the first element.

From now on, Max($\preceq$) (resp. Min($\preceq$)) denotes the set of maximal (resp. minimal) elements of a preordered set $(X, \preceq)$, and Fix(f) (resp. Per(f)) denotes the set of all fixed (resp. periodic) points of a map $f : X \to X$.

We obtained the following in [12,15]:

Brøndsted-Jachymski Principle. *Let $(X, \preceq)$ be a partially ordered set and $f : X \to X$ be a progressive map (that is, $x \preceq f(x)$ for all $x \in X$). Then we have*

$$\mathrm{Max}(\preceq) \subset \mathrm{Fix}(f) = \mathrm{Per}(f).$$

Similarly, if $f : X \to X$ is a anti-progressive map (that is, $f(x) \preceq x$ for all $x \in X$), then

$$\mathrm{Min}(\preceq) \subset \mathrm{Fix}(f) = \mathrm{Per}(f).$$

Let $(X, \preceq)$ be a partially ordered set and

$$S_+(x) := \{y \in X : x \preceq y\} \ \ (\text{resp. } S_-(x) := \{y \in X : y \preceq x\})$$

for any $x \in X$. Then we have the following:

Theorem 1. *Let $(X, \preceq)$ be a partially ordered set, $x_0 \in X$ such that $(S_+(x_0), \preceq)$ (resp. $(S_-(x_0), \preceq)$) has an upper bound $v \in S_+(x_0)$ (resp. a lower bound $v \in S_-(x_0)$).*

Then the following statements hold:

(1) *v is a maximal (resp. minimal) element, that is, $v \not\preceq w$ (resp. $w \not\preceq v$) for all $w \in X \backslash \{v\}$.*

(2) *For each chain C in $S_+(v)$ (resp. $S_-(v)$), we have $\bigcap_{x \in C} S_+(x) \neq \emptyset$ (resp. $\bigcap_{x \in C} S_-(x) \neq \emptyset$).*

(3) *Any map $f : X \to X$ satisfying $x \preceq f(x)$ (resp. $f(x) \preceq x$) for all $x \in S_+(x_0)$ (resp. $x \in S_-(x_0)$) has a fixed point $v \in S_+(x_0)$ (resp. $v \in S_-(x_0)$) and*

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preceq) \neq \emptyset \ \ (\textit{resp. } \mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Min}(\preceq) \neq \emptyset).$$

Proof. It suffices to prove the maximal case only.

(1) For any $w \in X \backslash \{v\}$, if $v \preceq w$, then $x_0 \preceq v \preceq w$, that is, $w \in S_+(x_0)$. Since v is an upper bound of $S_+(x_0)$, we have $w \preceq v$. Hence $v = w$, a contradiction. Therefore $v \not\preceq w$ and v is a maximal element of X.

(2) For the maximal $v \in S_+(x_0)$ in (1), we have $C = \{v\}$ is the unique chain in $S_+(v)$ and $\bigcap_{x \in C} S_+(x) = S_+(v) \neq \emptyset$, which proves (2).

(3) Since the maximal $v \in S_+(x_0)$ and $v \preceq f(v)$, we have $f(v) \in S_+(x_0)$. Therefore, $v = f(v)$ by the antisymmetry of $\preceq$. Now the conclusion holds by the Brøndsted-Jachymski Principle. □

Remark 2. (1) In Theorem 1, actually (1)-(3) are equivalent; see Metatheorem* in [21,22].

(2) For the motivation of Theorem 1 and its proof, we have a long story as shown in [22]. For the origin of maximal cases of statements (2) and (3), see ([26], Theorem 5.1) and ([22], Theorem 5.1*).

Note that (3) implies the following Zermelo type fixed point theorem:

Theorem 3. *For every partially ordered set $(X, \preceq)$ if every well-ordered subset has a least upper bound then every progressive map $f : X \to X$ has a fixed point and*

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preceq) \neq \emptyset.$$

See Zermelo [27] in 1908, Abian [28] in 1980, and Manka [29] in 1988. Theorem 3 is equivalently formulated in [13].

Recall that a fundamental fixed point theorem of Zermelo (see, e.g., Dunford-Schwartz ([30], p.5) says that

Proposition 4. *If $(X, \preceq)$ is a partially ordered set in which every chain has a supremum and a selfmap $f : X \to X$ is progressive, then f has a fixed point.*

This was given implicitly in Zermelo [27] in 1908 and formulated by Bourbaki [31] in 1949-50. Later Amann in 1977 derived several fixed point theorems from Proposition 4. For example, Tarski's fixed point theorem, fixed point theorems for condensing maps and nonexpansive maps.

Jachymski [32] in 2001 noted: "Under the Axiom of Choice, the assumption of Proposition 4 can be weakened to "each nonempty well-ordered subset has an upper bound. This improves Kneser's fixed point theorem [33] in 1950, which turns out to be equivalent to the Axiom of Choice as shown by Abian [34] in 1985."

Proposition 5. *Let $(X, \preccurlyeq)$ be a partially ordered set in which each nonempty well-ordered subset has a supremum. Then every progressive map $f : X \to X$ has a fixed point.*

Propositions 4 and 5 are consequences of our Theorem 3 and their conclusions can be improved to $\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preceq) \neq \emptyset$.

According to Toyoda [35] in 2021-22, "the Zermelo fixed point theorem is also known as the Bourbaki fixed point theorem or the Bourbaki-Kneser fixed point theorem. It implies the Bernstein-Cantor-Schröder theorem, the Caristi fixed point theorem, the Ekeland variational principle, the Takahashi minimization theorem, Nadler's fixed point theorem, and others. Moreover, under the Axiom of Choice, it implies Zorn's Lemma."

3 Extended Zorn's Lemma

A partially ordered set $(X, \preceq)$ is said to be *inductive* (resp. *complete*) if every non-empty chain in X has an upper bound (resp. a least upper bound).

The following was given in [15]:

Theorem 6. *Let $(X, \preceq)$ be a partially ordered set satisfying one of the following:*

(a) *a nonempty chain in X has an upper bound ($\Leftarrow$ X is inductive),*
(b) *a nonempty chain in X has a least upper bound ($\Leftarrow$ X is complete),*
(c) *a nonempty well-ordered subset of X has an upper bound,*
(d) *a nonempty well-ordered subset of X has a least upper bound,*

Then there exists a maximal element $v \in X$, that is, $v \not\preceq w$ for any $w \in X \backslash \{v\}$.

From the Brøndsted-Jachymski Principle and Theorem 6, we have the following improvement of Zorn's Lemma:

Theorem 7. *Let $(X, \preceq)$ be a partially ordered set satisfying one of (a)-(d) in* Theorem 6. *If $f : X \to X$ is progressive, then we have*

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preceq) \neq \emptyset.$$

Dual statements of Theorems 6 and 7 for the minimal case also hold.

The following is an example of Theorems 6 and 7, but Zorn's Lemma can not be applicable.

Example 8. Let $C = [0, 1] \times \{1\}$ and $D = \mathbb{R} \times \{0\}$ in $\mathbb{R}^2$ with their natural orders. Let $X = C \cup D$ be the partially ordered set and $f : X \to X$ a progressive map defined by

$$f(x, y) = \begin{cases} (\frac{1}{2}(x + 1), 1) & \text{if } (x, y) = (x, 1) \in C; \\ (x + 1, 0) & \text{if } (x, y) = (x, 0) \in D. \end{cases} \tag{1}$$

Then the chain or totally ordered subset $A = S_+(0, 1)$ has an upper bound or a supremum (1,1) $\in A$, which is a maximal element and a fixed point of a progressive map f. Note that any progressive map on C has a fixed point and that the chain D does not have any upper bound.

4 Generalized Caristi Fixed Point Theorem

The following is well-known by Caristi [4] in 1976:

Theorem 9. (Caristi) *If (X, ρ) is a complete metric space and $\phi : X \to \mathbb{R}^+$ lower semi-continuous, then in the Brøndsted order ($x \preceq y$ iff $\rho(x, y) \leq \phi(x) - \phi(y)$) every progressive map $f : X \to X$ has a fixed point.*

Recall that Kirk-Saliga [36] in 2001 and Chen-Cho-Yang [23] in 2002 introduced the following concept of lower semicontinuity from above:

Definition 10. [23] Let X be a metric space. A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *lower semicontinuous from above* if, for any point $x \in X$, $x_n \to x$ as $n \to \infty$ and $f(x_1) \geq f(x_2) \geq \cdots \geq f(x_n) \geq \cdots$ imply $\lim_{n\to\infty} f(x_n) \geq f(x)$.

Obviously, at any point, the usual lower semicontinuity implies lower semicontinuity from above, but the converse does not hold. In fact, Chen-Cho-Yang [23] gave an example of a function which is lower semicontinuous from above at a point, but not lower semicontinuous at that point.

Recall the following due to Kirk-Saliga [36] and Chen-Cho-Yang [23]:

Theorem 11. (Caristi's Fixed Point Theorem) *Let (D, d) be a complete metric space and a function $\phi : D \to \mathbb{R}^+$ be lower semi-continuous from above. Suppose that a mapping $f : D \to D$ satisfies the following:*

$$d(x, f(x)) \leq \phi(x) - \phi(f(x)) \quad \textit{for all } x \in D.$$

Then there exists $x_0 \in D$ such that $f(x_0) = x_0$.

Note that (D, d) can be made into a partially ordered set by defining

$$x \preceq y \iff \phi(y) \leq \phi(x)$$

for $x,\ y \in D$.

Here we give a new proof of the Caristi Theorem 11 due to Kirk-Saliga [36] and Chen-Cho-Yang [23]:

PROOF. Since $\phi : D \to \mathbb{R}^+$ is l.s.c. from above at any $z \in D$, for any $\{x_n\}$ converging to z such that

$$\phi(x_1) \geq \phi(x_2) \geq \cdots \geq \phi(x_n) \geq \cdots \Longrightarrow \lim_{n\to\infty} \phi(x_n) \geq \phi(z)$$

and hence $x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots \preceq z$. Note that $C = \{z\} \subset S_+(x_1)$ is a chain in $S_+(z)$. Let $v = z \in C$. Then $C = \{v\} \subset \bigcup_{x\in C} S_+(x) \neq \emptyset$. Hence, Theorem 1(2) holds, v is maximal by (1), and our Caristi theorem (3) holds in Theorem 1. □

The original Caristi theorem is equivalent to Zorn's Lemma. For its earlier proofs, see Kirk [37]. However, the extended version, Theorem 11, has an elementary proof as above. For, further generalizations of the Caristi theorem, Cobzaş [38] is a rich source of information.

5 Dual of Caristi Fixed Point Theorem

Until now, certain results are related to the maximality. We can obtain their dual formulations for the minimality. In this section, we obtain dual forms of the Caristi theorem.

We define the following motivated by Lin-Du [24]:

Definition 10.* Let X be a metric space. A function $f : X \to \overline{\mathbb{R}}$ is said to be *upper semicontinuous from below* if, for any point $x \in X$, $x_n \to x$ as $n \to \infty$ and $f(x_1) \leq f(x_2) \leq \cdots \leq f(x_n) \leq \cdots$ imply $\lim_{n\to\infty} f(x_n) \leq f(x)$.

The following is a dual of Theorem 11; see also [18].

Theorem 11.* *Let $(X, \preceq)$ be a partially ordered complete metric space, and a function $\varphi : X \to \mathbb{R}^+$ be upper semicontinuous from below and bounded from above such that*

$$y \preceq x \quad \textit{iff} \quad d(x,y) \leq \varphi(y) - \varphi(x) \quad \textit{for} \quad x, y \in X.$$

Then the dual of Theorem 11 hold, that is,

(α) *There exists a minimal element $v \in X$; that is, $w \not\preceq v$ for any $w \in X \backslash \{v\}$.*

Theorem 11* is the dual to the Caristi fixed point theorem 11 and can be stated as follows:

Theorem 12. *Let $(X, \preceq)$ be a partially ordered complete metric space, and a function $\varphi : X \to \mathbb{R}^+$ be upper semicontinuous from below and bounded from above such that*

$$y \preceq x \quad \textit{iff} \quad d(x,y) \leq \varphi(y) - \varphi(x) \quad \textit{for} \quad x, y \in X.$$

Then every anti-progressive map $f : X \to X$ has

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Min}(\preceq) \neq \emptyset.$$

Note that there are nearly one thousand papers related to the Caristi theorem for its extensions, modifications, and applications. However, this article has something different to them.

6 Unified Generalizations

All of the key results in the above can be unified by applying our Metatheorem and its variants. In fact, from our new 2023 Metatheorem in [21], we deduced the following prototype of Extreme Element Principles for non-empty valued multimaps:

Theorem 13. *Let $(X, \preceq)$ be a preordered set and A be a nonempty subset of X. Then the following statements are equivalent:*

(α) *There exists a maximal (resp. minimal) element $v \in A$, that is, $v \not\preceq w$ (resp. $w \not\preceq v$) for any $w \in X \backslash \{v\}$.*

(β) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ such that, for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X \backslash \{x\}$ satisfying $x \preceq y$ (resp. $y \preceq x$), then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(γ) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ satisfying $x \preceq f(x)$ (resp. $f(x) \preceq x$) for all $x \in A$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(δ) *Let $\mathfrak{F}$ be a family of multimaps $F : A \multimap X$ such that, for any $x \in A \backslash F(x)$, there exists $y \in X \backslash \{x\}$ satisfying $x \preceq y$ (resp. $y \preceq x$). Then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v \in F(v)$ for all $F \in \mathfrak{F}$.*

(ϵ) *If $\mathfrak{F}$ is a family of multimaps $F : A \multimap X$ such that $x \preceq y$ (resp. $y \preceq x$) holds for any $x \in A$ and any $y \in F(x) \backslash \{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = F(v)$ for all $F \in \mathfrak{F}$.*

(η) *If Y is a subset of X such that, for each $x \in A \backslash Y$, there exists a $z \in X \backslash \{x\}$ satisfying $x \preceq z$ (resp. $z \preceq x$), then there exists a $v \in A \cap Y$.*

Remark 14. (1) When $\mathfrak{F}$ is a singleton, $(\beta) - (\epsilon)$ are denoted by $(\beta 1) - (\epsilon 1)$, respectively, and these are also equivalent to $(\alpha) - (\eta)$. Hence, Theorem 13 implies 10 equivalent statements.

(2) Note that, in Theorem 13, $(\alpha) \Longleftrightarrow (\gamma 1)$ implies the Brøndsted-Jachymski Principle.

For multimaps permitting *empty values*, we derive the following Extreme Element Principle from the old 2023 Metatheorem:

Theorem 13*. *Let $(X, \preceq)$ be a preordered set and A be a nonempty subset of X. Then the following statements are equivalent:*

(α) *There exists a maximal (resp. minimal) element $v \in A$, that is, $v \not\preceq w$ (resp. $w \not\preceq v$) for any $w \in X \backslash \{v\}$.*

($\zeta 1$) *If a multimap $F : A \multimap X$ such that, for all $x \in A$ with set$F(x) \neq$, there exists $y \in X \backslash \{x\}$ satisfying $x \preceq y$ (resp. $y \preceq x$), then there exists $v \in A$ such that $F(v) = \emptyset$.*

($\zeta 2$) *Let $\mathfrak{F}$ be a family of multimaps $F : A \multimap X$ such that, for all $x \in A$ with $F(x) \neq \emptyset$, there exists $y \in X \backslash \{x\}$ satisfying $x \preceq y$ (resp. $y \preceq x$). Then there exists $v \in A$ such that $F(v) = \emptyset$ for all $F \in \mathfrak{F}$.*

From Theorem 13($\gamma 1$), we can deduce many examples of maps $f : X \to X$ satisfying $\text{Per}(f) = \text{Fix}(f) \neq \emptyset$; see [20]. Such sets X can have more rich properties by the following main theorem of Jachymski ([25], Theorem 2):

Theorem 15. [25] *Let X be a nonempty abstract set and $f : X \to X$. The following statements are equivalent:*

(a) $\text{Per}(f) = \text{Fix}(f) \neq \emptyset$.

(b) (Zermelo) *There exists a partial order* $\preceq$ *such that every chain in* $(X, \preceq)$ *has a supremum and* f *is progressive with respect to* $\preceq$.

(c) (Caristi) *There exists a complete metric* d *and a lower semicontinuous function* $\varphi : X \to \mathbb{R}^+$ *such that* f *satisfies Caristi's condition.*

(d) *There exists a complete metric* d *and a d-Lipschitzian function* $\varphi : X \to \mathbb{R}^+$ *such that* f *satisfies Caristi's condition and* f *is nonexpansive with respect to* d*; i.e.*

$$d(fx, fy) \leq d(x, y) \quad \textit{for all} \quad x, y \in X.$$

(e) (Hicks-Rhoades) *For each* $\alpha \in (0, 1)$*, there exists a complete metric* d *such that* f *is nonexpansive with respect to* d *and*

$$d(fx, f^2x) \leq \alpha d(x, fx) \quad \textit{for all} \quad x \in X.$$

(f) *There exists a complete metric* d *such that* f *is continuous with respect to* d *and for each* $x \in X$*, the sequence* $(f^n x)_{n=1}^{\infty}$ *is convergent (the limit may depend on* x*).*

(g) *There exists a partition of* X*,* $X = \bigcup_{\gamma \in \Gamma} X_\gamma$*, such that all the sets* X_γ *are nonempty, f-invariant and pairwise disjoint, and for all* $\gamma \in \Gamma$*,* $f|_{X_\gamma}$ *has a unique periodic point.*

(h) *For each* $\alpha \in (0, 1)$*, there exists a partition of* X*,* $X = \bigcup_{\gamma \in \Gamma} X_\gamma$*, and complete metrics* d_γ *on* X_γ *such that all the sets* X_γ *are nonempty; f-invariant and pairwise disjoint; and*

$$d_\gamma(fx, fy) \leq \alpha d_\gamma(x, y) \quad \textit{for all } x, y \in X.$$

From Theorem 15, we obtained the following useful theorem in [15]:

Theorem 16. *Let* $(X, \preceq)$ *be a nonempty partially ordered set and* $f : X \to X$*. The following statements are equivalent:*

(a) $\text{Per}(f) = \text{Fix}(f) \supset \text{Max}(\preceq) \neq \emptyset$.

(b) (Zermelo) *There exists an* $x_0 \in X$ *such that* $S(x_0) = \{x \in X : x_0 \preceq x\}$ *has an upper bound and* f *is progressive with respect to* $\preceq$.

(c) (Caristi) *There exists a complete metric* d *and a lower semicontinuous function* $\varphi : X \to \mathbb{R}^+$ *from above such that* f *satisfies the Caristi condition*

$$x \preccurlyeq f(x) \Longleftrightarrow d(x, f(x)) \leq \varphi(x) - \varphi(f(x))$$

for all $x \in X$.

Note that this theorem implies several of the Caristi type and the Zermelo type theorems in this article; see also [11-22].

7 Ekeland Principle

In order to obtain some equivalents of the well-known central result of Ivar Ekeland [11-3] on the variational principle for approximate solutions of minimization problems, we obtained a Metatheorem in [5-10] and related works in 1983-2000. Later in 2022 we found an extended version of the Metatheorem and, finally, the 2023 version in [15]. Our Theorem 13 can be applied to give equivalencies for various situations as we have shown in our previous works. Motivated by this, we derive the following; see also [20]:

Theorem 17. *Let (X, d) be a complete metric space and a proper function $\varphi : X \to \overline{\mathbb{R}}$ l.s.c. from above and bounded from below (resp. u.s.c. from below and bounded from above). Let $A = \operatorname{dom}\varphi = \{x \in X : -\infty < \varphi(x) < \infty\}$.*

Then the following equivalent statements hold:

(α) *There exists a 'maximal' (resp. 'minimal') element $v \in A$, that is,*

$$d(v, w) > \varphi(v) - \varphi(w) \quad (resp.\ \ d(v, w) > \varphi(w) - \varphi(v))$$

for any $w \in X\backslash\{v\}$.

(β) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ such that, for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X\backslash\{x\}$ satisfying*

$$d(x, y) \leq \varphi(x) - \varphi(y) \quad (resp.\ \ d(x, y) \leq \varphi(y) - \varphi(x)),$$

then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.

(γ) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ satisfying*

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)) \quad (resp.\ \ d(x, f(x)) \leq \varphi(f(x)) - \varphi(x))$$

for all $x \in A\backslash\{f(x)\}$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.

(δ) *Let $\mathfrak{F}$ be a family of multimaps $T : A \multimap X$ such that, for any $x \in A\backslash T(x)$, there exists $y \in X\backslash\{x\}$ satisfying*

$$d(x, y) \leq \varphi(x) - \varphi(y) \quad (resp.\ \ d(x, y) \leq \varphi(y) - \varphi(x)),$$

then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v \in T(v)$ for all $T \in \mathfrak{F}$.

(ϵ) *If $\mathfrak{F}$ is a family of multimaps $T : A \multimap X$ such that*

$$d(x, y) \leq \varphi(x) - \varphi(y) \quad (resp.\ \ d(x, y) \leq \varphi(y) - \varphi(x))$$

holds for any $x \in A$ and any $y \in T(x)\backslash\{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = T(v)$ for all $T \in \mathfrak{F}$.

(η) *If Y is a subset of X such that, for each $x \in A\backslash Y$, there exists a $z \in X\backslash\{x\}$ satisfying*

$$d(x, z) \leq \varphi(x) - \varphi(z) \quad (resp.\ \ d(x, z) \leq \varphi(z) - \varphi(x)),$$

then there exists a $v \in A \cap Y$.

($\theta 1$) *There exists $v \in A$ such that, for each chain C in $S(v)$, we have $\bigcap_{x\in C} S(x) \neq \emptyset$.*

($\theta 2$) *There exist $v \in A$ and a maximal chain C^* in $S(v)$, we have $\bigcap_{x\in C^*} S(x) \neq \emptyset$.*

In Theorem 16, (X, d) can be made into a partially ordered set $(X, \preceq)$ by defining

$$x \preccurlyeq y \iff \varphi(y) \leq \varphi(x) \quad (\text{resp. } x \preceq y \iff \varphi(x) \leq \varphi(y))$$

for $x,\ y \in X$.

Theorem 17 includes various earlier results and is very useful as shown in [20]. Especially, (α) for maximal case is called the *weak Ekeland Principle* and that $(\gamma 1)$ extends the Caristi theorem.

In 1989, by using Ekeland's variational principle, Mizoguchi-Takahashi [39] derived the following Caristi-Kirk theorem [40], which is the set-valued version of the Caristi fixed point theorem:

Theorem 18. (Caristi-Kirk) *Let (X, d) be a complete metric space and $T : X \multimap X$ be a multimap with nonempty values such that for each $x \in X$, there exists $y \in T(x)$ satisfying $d(x, y) + \varphi(y) \leq \varphi(x)$, where $\varphi : X \to \mathbb{R} \cup \{\infty\}$ is a proper, lower semicontinuous and bounded below functional. Then, T has a fixed point, that is, there exists $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$.*

Here the lower semicontinuous function can be replaced by the one *from above*. Moreover, so does the following extended form of the Takahashi Principle [41-45]:

Theorem 19. (Takahashi) *Let (X, d) be a complete metric space and $\varphi : X \to \mathbb{R} \cup \{\infty\}$ a proper function lower semicontinuous from above and bounded from below. If for every $x \in \operatorname{dom} \varphi$ with $\varphi(x) > \inf \varphi(X)$ there exists an element $y \in \operatorname{dom} \varphi \backslash \{x\}$ such that $\varphi(y) + d(x, y) \leq \varphi(x)$, then φ attains its minimum on X, i.e., there exists $z \in \operatorname{dom} \varphi$ such that $\varphi(z) = \inf \varphi(X)$.*

Further, from Theorem 17$(\alpha), (\gamma)$ and the Brøndsted-Jachymski Principle, we have the following:

Theorem 20. *Let (X, d) be a complete metric space and $\varphi : X \to \mathbb{R} \cup \{\infty\}$ a proper function l.s.c. from above and bounded from below (resp. u.s.c. from below and bounded from above). If $f : X \to X$ is a map such that*

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)) \quad (\textit{resp. } d(x, f(x)) \leq \varphi(f(x)) - \varphi(x))$$

for any $x \in X$. Then we have

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preceq) \neq \emptyset \quad (\textit{resp. } \mathrm{Fix}(f) = \mathrm{Per}(f)) \supset \mathrm{Min}(\preceq) \neq \emptyset.$$

Consequently, this section demonstrates the usefulness of our Metatheorem. Until now, we gave more than one hundred examples or applications of our Metatheorem, and each of them might have useful consequences.

8 Analysis of Applications of Principles

In this section, we analyze certain situations in the present article such that Theorems 1 and 13 are applicable:

(1) (Zermelo type) Let $x_0 \in X$ and $A = S_+(x_0)$ (resp. $A = S_-(x_0)$) have an upper bound, or more general assumptions.

Theorems 1, 3, Propositions 4, 5 and Theorem 15 belong to this case.

The Zermelo fixed point theorem implies the Caristi fixed point theorem, the Bernstein-Cantor-Schröder theorem, the Ekeland variational principle, the Takahashi minimization theorem, and others. Moreover, under the Axiom of Choice, it implies Zorn's Lemma.

(2) (Zorn type) Let $X = A$ satisfy one of (a)-(d) in Theorem 6.

Theorems 6, 7, and their minimal cases belong to this case.

The 29 references of [8] show the origins, variants, consequences, applications of a particular form of Theorem 6, and we will only indicate names of their authors like Abian (1971), Ekeland [3](1979), Bishop-Phelps (1961), Turinici (1980-1984), Smithon (1971, 1973), Hoft-Hoft (1976), Tuy (1981), Kasahara (1975), Maschler-Peleg (1976), Phelps (1964), Caristi [4](1976), Banach (1922), and others. Recall that Tasković (1986) showed an equivalent form of Zorn's lemma.

(3) (Caristi type) Let (X, d) be a complete metric space and a function $\phi : X \to \mathbb{R}^+$ be lower semicontinuous from above. Define partial order on X using ϕ.

Examples are Theorems 9, 11, 15, 16 and 18.

Equivalent formulations of the Caristi theorem were originally given in Park [5, 6] in 1985-1986. The Caristi theorem implies the Banach contraction principle and numerous applications.

(4) (dual Caristi type) Let (X, d) be a complete metric space and a function $\phi : X \to \mathbb{R}^+$ be upper semicontinuous from below. Define partial order on X using ϕ.

Theorems 11* and 12 are examples of this case.
Extensions of the dual Caristi theorem were given by Lin-Du [24] and their equivalent formulations by Park [12,20].

(5) (Ekeland type) Let (X, d) be a complete metric space and $\varphi : X \to \mathbb{R} \cup \{\infty\}$ a proper function l.s.c. from above and bounded from below (resp. u.s.c. from below and bounded from above). Define partial order on X using ϕ.

Theorems 17–20 are examples of this case.

The original Theorem 17 was given in [5, 6] in 1985-1986. It implies the variational principle of Ekeland (1979), works of Tuy (1981), Kasahara (1975), Mascher-Peleg (1976), etc. Classical applications of Theorem 17 are numerous in vast fields of mathematical sciences.

By applying Theorems 1 and 13 to each of such types, we can obtain more than one hundred true statements. Only some of them are known as famous theorems; see our previous works in the references. We will not trace all of them.

9 Epilogue

As we have seen on Metatheorem [15], the maximal elements in certain preordered sets can be reformulated to fixed points or stationary points of maps or multimaps and to common fixed points or common stationary points of a family of maps or multimaps, and conversely. Actually such points are same as we have seen in the proof of Metatheorem. Therefore, if we have a theorem on any of such points, it can be converted to at least nine equivalent theorems on other types of points without any serious argument. Some authors seem to be not recognized this fact yet. Its applications are numerous.

In many fields of mathematical sciences, there are plentiful number of theorems concerning maximal points or various fixed points that can be applicable our Metatheorem. Some of such theorems can be seen in our previous works and the present article. Therefore, a metatheorem like Theorems 12 and 14 are machines to expand our knowledge easily. In this article we presented relatively old and well-known examples.

Since 2022, we have published many articles applying our Metatheorem and its variants. We hope the reader's engagement to find more applications of them.

Competing Interests

Author has declared that no competing interests exist.

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