Application of Adomian Decomposition Method to Solving Higher Order Singular Value Problems for Ordinary Differential Equations

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Authors contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Original Research Article

Abstract

This paper is an attempt to solve singular value problems for higher order ordinary differential equation by using new modification of Adomian Decomposition Method (ADM). Convergent series solution of considered problem have been obtained. Three numerical examples are discussed to validate the strength and ease of the method used.

1 Introduction

The goal of this paper is to study the numerical solution of singular initial value problems of higher order. Solving singular equations with initial or boundary conditions is an important issue in science application. Therefore, in the recent studies, many researchers have been studying singular

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equations and introducing many methods to solve this type of equations [1-5]. One of those methods is Adomian Decomposition method (ADM) [6-12], which is a numerical method that was introduced by George Adomian in 1980s [13,14] to solve different equations. This method can be used to solve differential equations with integer or fractional order[15,16], ordinary or partial, with initial value or boundary problems, with variable or constant coefficients, linear or nonlinear, homogeneous or non homogeneous. The major difficulty for solving the model under study is the presence of singularity so we submit a novel modified of ADM which would enable us to establish an useful solution to the singular initial value problems via this method.

2   Structure of The Singular Equations of Higher Order

In the section we present generalized formula to derive various kinds of the singular equations of different order for various value of α, the generalized formula given in the form

$$x^{-1}\frac{d^m}{dx^m}e^{-\alpha x}\frac{d}{dx}e^{\alpha x}xy = g(x,y), \tag{2.1}$$

for $m = 1$, Eqs. (1) becomes the first kind of singular equation of second-order [4] given as

$$y'' + \frac{2+\alpha x}{x}y' + \frac{\alpha}{x}y = g(x,y), \tag{2.2}$$

for $m = 2$, Eqs. (1) becomes the second kind of singular equation of third-order given as

$$y''' + \frac{3+\alpha x}{x}y'' + \frac{2\alpha}{x}y' = g(x,y), \tag{2.3}$$

for $m = 3$, Eqs. (1) becomes the third kind of singular equation of fourth-order given as

$$y^{(4)} + \frac{4+\alpha x}{x}y''' + \frac{3\alpha}{x}y'' = g(x,y), \tag{2.4}$$

and so on, we have

$$y^{(k)} + \frac{k+\alpha x}{x}y^{(k-1)} + \frac{(k-1)\alpha}{x}y^{(k-2)} = g(x,y). \tag{2.5}$$

3   Adomian Decomposition Method

In this part we will show the basic solution steps for this method that we will use in this paper to solve this type of equations (5) with the following initial conditions
$y(0) = a_1, y'(0) = a_2, y''(0) = a_3, ...y^{(k)}(0) = a_k$.
According to the ADM we rewrite Eq.(1) in operator form as

$$Ly = g(x,y). \tag{3.1}$$

The new inverse integral operator L^{-1} is given by

$$L^{-1}(.) = x^{-1}e^{-\alpha x}\int_0^x e^{\alpha x}\underbrace{\int_0^x\int_0^x...\int_0^x}_{k-1} x(.)dxdx...dx. \tag{3.2}$$

Applying L^{-1} on both sides of Eq.(6) we have

$$y = \gamma(x) + L^{-1}g(x,y), \tag{3.3}$$

such that

$$L\gamma(x) = 0.$$

The ADM assumes a series solution for $y(x)$ given by

$$y(x) = \sum_{n=0}^{\infty} y_n(x),$$

(3.4)

and the nonlinear terms defined by the series

$$g(x, y) = \sum_{n=0}^{\infty} A_n,$$

(3.5)

where the components $y_n(x)$ of the solution $y(x)$ will be determined recurrently, and the A_n are the Adomian polynomials, specific algorithms were seen in [2] to formulate Adomian polynomials . The flowing algorithm:

$$A_0 = G(y_0),$$
$$A_1 = G'(y_0)y_1,$$
$$A_2 = G'(y_0)y_{(2)} + \frac{1}{2}G''(y_0)y_1^2,$$
$$A_3 = G'(y_o)y_{(3)} + G''(y_0)y_1y_2 + \frac{1}{3!}G'''(y_0)y_1^3,$$

(3.6)

$$\cdots$$

from (8) , (9) and (10) we have

$$\sum_{n=0}^{\infty} y_{(n)} = \gamma(x) + L^{-1} \sum_{n=0}^{\infty} A_n.$$

(3.7)

To determine the components $y_n(x)$,we use ADM by using the relation

$$y_0 = \gamma(x) + L^{-1}f(x),$$
$$y_{n+1} = -L^{-1}A_n, \ n \geq 0.$$

(3.8)

4 Numerical Examples

Example 1. Consider the problem

$$y^{(4)} + (\frac{4+8x}{x})y''' + \frac{24}{x}y'' = (30 + 120\,x - 171\,x^4 + 72\,x^5 + 30\,x^8 - 48\,x^9)y^{-15},$$

(4.1)

$y(0) = 1, y'(0) = 0, y''(0) = 0, y'''(0) = 0,$
We define the operator as

$$L(.) = x^{-1}\frac{d^3}{dx^3}e^{-8x}\frac{d}{dx}e^{8x}x(.),$$

the inverse operator L^{-1} is given by

$$L^{-1}(.) = x^{-1}e^{-8x}\int_0^x e^{8x}\int_0^x \int_0^x \int_0^x x(.)dxdxdxdx.$$

Eq.(14) can be written as operator form

$$Ly = (30 + 120\,x - 171\,x^4 + 72\,x^5 + 30\,x^8 - 48\,x^9)y^{-15}.$$

(4.2)

Taking L^{-1} on both sides of (15) and using the condition gives

$$y(x) = 1 + L^{-1}(30 + 120\,x - 171\,x^4 + 72\,x^5 + 30\,x^8 - 48\,x^9)y^{-15}.$$

To get the solution we use the iterative formula

$$y_0 = 1,$$

$$y_{n+1} = L^{-1}(30 + 120\,x - 171\,x^4 + 72\,x^5 + 30\,x^8 - 48\,x^9)A_n, \ n \geq 0, \tag{4.3}$$

where the nonlinear term y^{-15} has Adomian polynomials A_n as the following

$$A_0 = y_0^{-15},$$

$$A_1 = -15y_1 y_0^{-16}, \tag{4.4}$$

$$A_2 = -15y_2 y_0^{-16} + 120y_1^2 y_0^{-17},$$

by substituting (17) into (16) we have the first few components of solution using ADM successively as follows

$$y_0(x) = 1,$$

$$y_1(x) = \frac{x^4}{4} - \frac{19\,x^8}{336} + \frac{5\,x^9}{84} - \frac{10\,x^{10}}{231} + \frac{20\,x^{11}}{693} - \frac{577\,x^{12}}{36036} + \cdots,$$

$$y_2(x) = \frac{-25\,x^8}{672} - \frac{5\,x^9}{84} + \frac{10\,x^{10}}{231} - \frac{20\,x^{11}}{693} + \frac{4663\,x^{12}}{82368} + \cdots$$

$$y_3(x) = \frac{1805\,x^{12}}{128128} + \cdots.$$

This gives the series solution

$$y(x) = 1 + \frac{x^4}{4} - \frac{3\,x^8}{32} + \frac{7\,x^{12}}{128} - \cdots.$$

Which is quite close to Taylor expansion of exact solution $y(x) = (1 + x^4)^{\frac{1}{4}}$.

Example 2. We consider the problem

$$y^{(5)} + (\frac{5 - 10x}{x})y''' - \frac{40}{x}y'' = \frac{4y\left(15 - 150\,x + 90\,x^2 - 200\,x^3 + 60\,x^4 - 40\,x^5 + 8\,x^6\right)}{x}, \tag{4.5}$$

$y(0) = 1, y'(0) = 0, y''(0) = 2, y'''(0) = 0, y^{(4)} = 0.$

We define the operator as

$$L(.) = x^{-1}\frac{d^4}{dx^4}e^{10x}\frac{d}{dx}e^{-10x}x(.),$$

the inverse operator L^{-1} is given by

$$L^{-1}(.) = x^{-1}e^{10x}\int_0^x e^{-10x}\int_0^x \int_0^x \int_0^x \int_0^x x(.)dxdxdxdxdx.$$

Eq.(18) can be written as operator form

$$Ly = \frac{4\left(15 - 150\,x + 90\,x^2 - 200\,x^3 + 60\,x^4 - 40\,x^5 + 8\,x^6\right)}{x}y. \tag{4.6}$$

Taking L^{-1} on both sides of (19) and using the condition gives

$$y(x) = 1 + x^2 + L^{-1}(\frac{4y\left(15 - 150\,x + 90\,x^2 - 200\,x^3 + 60\,x^4 - 40\,x^5 + 8\,x^6\right)}{x})$$

To get the solution we use the iterative formula

$$y_0 = 1 + x^2,$$

$$y_{n+1} = L^{-1} \frac{4 \left(15 - 150\,x + 90\,x^2 - 200\,x^3 + 60\,x^4 - 40\,x^5 + 8\,x^6\right)}{x} y_n, \ n \geq 0, \qquad (4.7)$$

we have the first few components of solution using ADM successively as follows

$$y_0(x) = 1 + x^2,$$

$$y_1(x) = \frac{x^4}{2} + \frac{x^6}{6} + \frac{5\,x^8}{126} + \frac{x^9}{126} + \frac{2\,x^{10}}{165} + \frac{5\,x^{11}}{594} + \frac{43\,x^{12}}{6435} + \frac{43\,x^{13}}{9009} + \frac{86\,x^{14}}{27027} + \ldots,$$

$$y_2(x) = \frac{x^8}{504} - \frac{x^9}{126} - \frac{x^{10}}{264} - \frac{5\,x^{11}}{594} - \frac{1717\,x^{12}}{324324} - \frac{21643\,x^{13}}{4540536} - \frac{86269\,x^{14}}{28814940} + \ldots$$

$$y_3(x) = \frac{x^{12}}{1297296} - \frac{29\,x^{13}}{4540536} + \frac{3089\,x^{14}}{299675376} + \ldots.$$

This gives the series solution

$$y(x) = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24} + \frac{x^{10}}{120} + \frac{x^{12}}{720} + \frac{x^{14}}{5040} + \ldots.$$

Which is quite close to Taylor expansion of exact solution $y(x) = e^{x^2}$.

Example 3. We consider the problem

$$y^{(4)} + (\frac{4 + 4x}{x})y''' + \frac{12}{x}y'' =$$
$$-8 \left(-15 - 30\,x + 129\,x^4 + 6\,x^5 - 49\,x^8 + 38\,x^9 - x^{12} + 2\,x^{13}\right) e^{4(1+x)}e^{-4y} \qquad (4.8)$$

$y(0) = y'(0) = 1, y''(0) = y'''(0) = 0,$

We define the operator as

$$L(.) = x^{-1} \frac{d^3}{dx^3} e^{-4x} \frac{d}{dx} e^{4x} x(.),$$

the inverse operator L^{-1} is given by

$$L^{-1}(.) = x^{-1}e^{-4x} \int_0^x e^{4x} \int_0^x \int_0^x \int_0^x x(.)dxdxdxdx.$$

Eq.(21) can be written as operator form

$$Ly = -8 \left(-15 - 30\,x + 129\,x^4 + 6\,x^5 - 49\,x^8 + 38\,x^9 - x^{12} + 2\,x^{13}\right) e^{4(1+x)}e^{-4y}. \qquad (4.9)$$

Taking L^{-1} on both sides of (22) and using the condition gives

$y(x)$

$$= 1 + x - 8L^{-1}\left(-15 - 30\,x + 129\,x^4 + 6\,x^5 - 49\,x^8 + 38\,x^9 - x^{12} + 2\,x^{13}\right) e^{4(1+x)}e^{-4y}.$$

To get the solution we use the iterative formula

$$y_0 = 1 + x,$$

$$y_{n+1} = -8L^{-1}\left(-15 - 30\,x + 129\,x^4 + 6\,x^5 - 49\,x^8 + 38\,x^9 - x^{12} + 2\,x^{13}\right) e^{4(1+x)})A_n, \qquad (4.10)$$

$$n \geq 0,$$

where the nonlinear term y^{-15} has Adomian polynomials A_n as the following

$$A_0 = e^{-4y_0},$$

$$A_1 = -4y_1 e^{-4y_0}, \qquad (4.11)$$

$$A_2 = (-4y_2 + 8y_1^2)e^{-4y_0},$$

by substituting (24) into (23) we have the first few components of solution using ADM successively as follows

$$y_0(x) = 1 + x,$$

$$y_1(x) = x^4 - \frac{43\,x^8}{126} + \frac{8\,x^9}{63} - \frac{32\,x^{10}}{693} + \frac{32\,x^{11}}{2079} + \frac{2447\,x^{12}}{135135} + \ldots,$$

$$y_2(x) = \frac{-10\,x^8}{63} - \frac{8\,x^9}{63} + \frac{32\,x^{10}}{693} - \frac{32\,x^{11}}{2079} + \frac{34438\,x^{12}}{135135} + \ldots$$

$$y_3(x) = \frac{544\,x^{12}}{9009} + \ldots.$$

This gives the series solution

$$y(x) = 1 + x + x^4 - \frac{x^8}{2} + \frac{x^{12}}{3} + \ldots.$$

Which is quite close to Taylor expansion of exact solution

$$y(x) = 1 + x + \log(1 + x^4).$$

5   Conclusion

The proposed ADM was employed for handling solutions of linear and nonlinear singular value problems with initial conditions. Our results establish that the method is really successful to find approximate solutions that have clearly approached the exact solutions of differential equations. This work demonstrated that this technique is reliable. We recommend that in future, some studies need to be done for solving fractional singular differential equations with boundary conditions by using ADM .

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Competing Interests

Authors have declared that no competing interests exist.

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