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Inverse Lomax-Exponentiated G (IL-EG) Family of Distributions: Properties and Applications

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Authors contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

A new generator of continuous distributions called the Inverse Lomax-Exponentiated G family, which has three extra positive parameters is proposed. The structural properties of the new family that holds for any continuous baseline model including explicit density function expressions, moments, inequality measurements, moment generating function, reliability functions, Renyi and Shanon entropies, and distribution of order statistics are derived. A Monte Carlo simulation to test the efficiency of the maximum likelihood estimates is conducted. The application of the new sub-model to the two data sets using the maximum likelihood method indicates that the new model is better than the existing competitors.

Keywords: Inverse Lomax exponentiated uniform distribution; Inverse Lomax exponentiated Weibull distribution; Inverse Lomax-G; Monte Carlo Simulation; Inverse Lomax exponentiated Burr III distribution; Inverse Lomax Exponential G Family.

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1 Introduction

Inverse Lomax (IL) is a part of a distribution of Beta form. Some of the families include Singh Maddala, Pareto, Log-logistics, Dagum, Generalized second-type beta distributions among others see ([1]). Since then, IL distribution has gained a lot of attention in many fields such as Actuarial Science and Economics (see [1]), Geophysical data (see [2]), Survival analysis (see [3], [4]), and Medical Science (see [5] and [6]).

There have been some attempts to define new families of probability distributions that improve well-known distribution families while at the same time providing greater flexibility in the realistic modeling of data. Following from the T-X approach by [7], we define the cumulative distribution function (cdf) as

$$J(x) = \int_{g}^{D(M(x;\upsilon))} r(f)df \tag{1.1}$$

where $D(M(x;\upsilon))$ is a cdf-function of $M(x;\upsilon)$ of any random variable (RV) X that $D(M(x;\upsilon)) = \frac{M(x;\upsilon)}{\bar{M}(x;\upsilon)} = \frac{M(x;\upsilon)}{1-M(x;\upsilon)}$ satisfies the conditions below:
(a) $D(M(x;\upsilon)) \in [g,h]$
(b) $D(M(x;\upsilon))$ is monotonically non-decreasing and differentiable
(c) $D(M(x;\upsilon)) \Rightarrow$ g as x $\Rightarrow -\infty$ and $D(M(x;\upsilon)) \Rightarrow h$ as $x \Rightarrow \infty$. Let F be a continuous random variable with pdf r(f) defined on [g, h].

Some of the generalized families of distributions based on this approach in the literature include Weibull G by [8], Lomax Generator of distributions by [9], Odd Generalized Exponential G by [10], Odd Lindley G family by [11], Gompertz G family by [12], Zubair G by [13], Odd Frechet G family by [14], Power Lindley G by [15], Topp Leone Exponentiated G by [16], Odd Chen G by [17], Kumaraswamy Odd Rayleigh G by [18], Burr X Exponential G by [19], and Inverse Lomax G by [20].

Some of the Exponentiated distributions in the literature include: Exponentiated-Weibull distribution by [21], Exponentiated-Gumbel distribution by [22], Exponentiated-Chen distribution by [23], the Exponentiated-New Weighted Weibull Distribution by [24], Exponentiated additive Weibull distribution by [25], the Lomax exponentiated Weibull model by [26] among others.

Inverse Lomax (IL) distribution has both scale β and shape θ parameters which makes it more flexible in modeling datasets. However, we wish to generalize the IL distribution, ostensibly to make it more flexible for wider application. The pdf and cdf of the IL distribution are given by

$$m(x;\theta,\beta) = \frac{\theta\beta}{x^2}\left(1+\frac{\beta}{x}\right)^{-(1+\theta)} \tag{1.2}$$

$$M(x;\theta,\beta) = \left(1+\frac{\beta}{x}\right)^{-(\theta)}; \qquad x>0,\theta,\beta>0 \tag{1.3}$$

The rest of the article is structured as follows. In Section 2, we defined the Inverse Lomax Exponentiated G Family. Some new models based on the IL-EG family are derived in section 3. Whereas Section 4 presents a mixture representation of the cdf, some of the mathematical properties of the Inverse Lomax exponentiated G (IL-EG) family including the reliability and inequality measures, quantile function, moments, moment generating function, order statistics and entropies are given in Section 5. The estimation of the parameters of the IL-EG family using the method of maximum likelihood follows in Section 6, The results of a Monte Carlo simulation study using the new Inverse Lomax Exponentiated Burr III (IL-EBIII) model are presented in Section 7. In Section 8, we applied the IL-EBIII to two real-world datasets and compared its performance with some existing distributions. Lastly, Section 9 concludes the paper.

2 The Inverse Lomax Exponentiated G (IL-EG) Family

In this section of the paper, we derived the Inverse Lomax Exponentiated G Family of distributions as well as the probability density function (pdf), cdf, hazard function (hf), reversed hazard function (rhf), survival function (sf), and cumulative hazard functions (H) were displayed.

Let $M(x;\upsilon)$ and $m(x;\upsilon)$ be the baseline cdf and pdf, and let $\vec{\zeta}$ be a vector of parameters i.e $\vec{\zeta} = (\theta, \beta, \upsilon, \lambda)^T$, let $r(f)$ be as defined in equation 1.2. Then we define the cdf $J(x;\zeta)$ of the IL-EG family of distributions as

$$J(x;\zeta) = \int_0^{\left[\frac{\bar{M}(x;\upsilon)}{M(x;\upsilon)}\right]^\lambda} r(f)df = \left(1+\beta\left[\frac{\bar{M}(x;\upsilon)}{M(x;\upsilon)}\right]^\lambda\right)^{-\theta}; \qquad x>0, \theta, \beta, \lambda, \upsilon > 0 \tag{2.1}$$

where θ, λ, and β are the three additional parameters. The corresponding pdf $j(x;\zeta)$ of IL-EG family is obtained by differentiating Equation 2.1 and is given below:

$$j(x;\zeta) = \frac{\theta\beta\lambda m(x;\upsilon)\bar{M}^{\lambda-1}(x;\upsilon)}{M^{\lambda+1}(x;\upsilon)}\left(1+\beta\left[\frac{\bar{M}(x;\upsilon)}{M(x;\upsilon)}\right]^\lambda\right)^{-(\theta+1)} \tag{2.2}$$

The hazard function (v), reversed hazard function (r), survival function (s), and cumulative hazard functions (K) are also presented below:

$$v(x;\zeta) = \frac{\theta\beta\lambda m(x;\upsilon)\bar{M}^{\lambda-1}(x;\upsilon)}{M^{\lambda+1}(x;\upsilon)}\left(1+\beta\left[\frac{\bar{M}(x;\upsilon)}{M(x;\upsilon)}\right]^\lambda\right)^{-(\theta+1)}\left[1-\left(1+\beta\left[\frac{\bar{M}(x;\upsilon)}{M(x;\upsilon)}\right]^\lambda\right)^{-\theta}\right]^{-1} \tag{2.3}$$

$$r(x;\zeta) = \frac{\theta\beta\lambda m(x;\upsilon)\bar{M}^{\lambda-1}(x;\upsilon)}{M^{\lambda+1}(x;\upsilon)\left(1+\beta\left[\frac{\bar{M}(x;\upsilon)}{M(x;\upsilon)}\right]^\lambda\right)} \tag{2.4}$$

$$s(x;\zeta) = 1-\left(1+\beta\left[\frac{\bar{M}(x;\upsilon)}{M(x;\upsilon)}\right]^\lambda\right)^{-\theta} \tag{2.5}$$

$$K(x;\zeta) = -log[s(x)] = -log\left[1-\left(1+\beta\left[\frac{\bar{M}(x;\upsilon)}{M(x;\upsilon)}\right]^\lambda\right)^{-\theta}\right] \tag{2.6}$$

The quantile function (qf) of IL-Exponetiated G family can be derived by inverting Equation 2.1 as follows

$$Q(U) = M^{-1}\left\{\frac{1}{1+\left[\frac{U^{-\frac{1}{\theta}}-1}{\beta}\right]^{\frac{1}{\lambda}}}\right\} \tag{2.7}$$

where $M^{-1}(.)$ is qf of the baseline distribution, U is uniformly distributed i.e $U \sim U(0,1)$, and Eqt. 2.7 can be used to draw samples from IL-EG family of distributions for purposes of Monte Carlo simulation studies.

3 Some IL-EG Sub-models

Here, we present three new sub-models of the IL-EG family of distributions: the Inverse Lomax-Exponentiated Uniform (IL-EU) distribution, the Inverse Lomax-Exponentiated Weibull (IL-EW), and the Inverselomax-Exponetiated Burr III distribution (IL-EBIII).

3.1 The IL-EU Model

Suppose that the parent distribution is Uniform on $(0, \tau)$. Then

$$m(x;\tau) = \frac{1}{\tau}, \quad \tau > 0 \qquad 0 < x < \tau < \infty$$

and

$$M(x;\tau) = \frac{x}{\tau}, \quad \tau > 0 \qquad 0 < x < \tau < \infty$$

Then, the Inverse Lomax Exponentiated Uniform (IL-EU) distribution has the cdf given by:

$$J_{IL-EU}(x;\theta,\beta,\lambda,\tau) = \left(1 + \beta\left[\left(\frac{x}{\tau}\right)^{-1} - 1\right]^{\lambda}\right)^{-\theta} \tag{3.1}$$

For $0 < x < \tau < \infty$
The corresponding pdf of Equation (3.1) is given by

$$j_{IL-EU}(x;\theta,\beta,\lambda,\tau) = \frac{\theta\beta\lambda\left[1 - \frac{x}{\tau}\right]^{\lambda-1}}{\tau\left[\frac{x}{\tau}\right]^{\lambda+1}}\left(1 + \beta\left[\left(\frac{x}{\tau}\right)^{-1} - 1\right]^{\lambda}\right)^{-(\theta+1)} \tag{3.2}$$

$0 < x < \tau < \infty$ and $\theta, \beta, \lambda > 0$ The v(x), K(x), and r(x) are given by

$$v_{IL-EU}(x;\theta,\beta,\lambda,\tau) = \frac{\theta\beta\lambda\left[1 - \frac{x}{\tau}\right]^{\lambda-1}}{\tau\left[\frac{x}{\tau}\right]^{\lambda+1}\left[1 - \left(1 + \beta\left[\left(\frac{x}{\tau}\right)^{-1} - 1\right]^{\lambda}\right)^{-\theta}\right]}\left(1 + \beta\left[\left(\frac{x}{\tau}\right)^{-1} - 1\right]^{\lambda}\right)^{-(\theta+1)} \tag{3.3}$$

$$K_{IL-EU}(x;\theta,\beta,\lambda,\tau) = -log\left[1 - \left(1 + \beta\left[\left(\frac{x}{\tau}\right)^{-1} - 1\right]^{\lambda}\right)^{-\theta}\right] \tag{3.4}$$

$$r_{IL-EU}(x;\theta,\beta,\lambda,\tau) = \frac{\theta\beta\lambda\left[1 - \frac{x}{\tau}\right]^{\lambda-1}}{\left(1 + \beta\left[\left(\frac{x}{\tau}\right)^{-1} - 1\right]^{\lambda}\right)\tau\left[\frac{x}{\tau}\right]^{\lambda+1}} \tag{3.5}$$

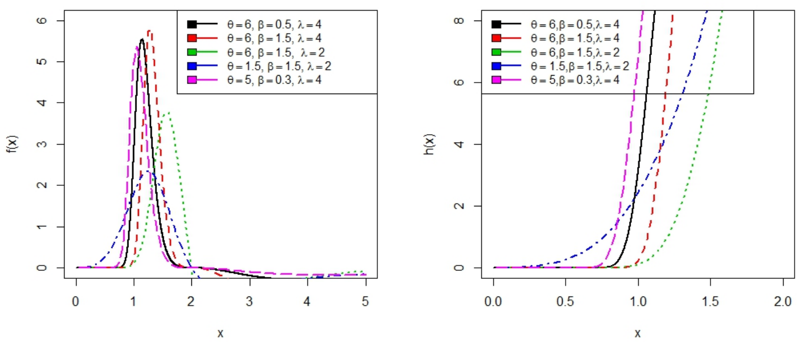

Fig. 1. Density and hazard rate plots of IL-EU distribution with fixed $\tau = 2$ and varying θ, β, and λ.

Figs. (1.) illustrates the various shapes of the density and hazard functions of the IL-EU using some selected parameter values. The density can be symmetric, J-shaped, and unimodal depending on the parameter values chosen. This includes J-shaped and non-decreasing.

3.2 The IL-EW Model

If the parent distribution is Weibull, then

$$m(x;\tau,\alpha) = \tau\alpha x^{\alpha-1}e^{\{-\tau x^{\alpha}\}}$$

and

$$M(x;\tau,\alpha) = 1 - e^{\{-\tau x^{\alpha}\}}$$

. With $x > 0$, $\tau, \alpha > 0$, respectively. Then the IL-EW distribution has the cdf given by:

$$J_{IL-EW}(x;\theta,\beta,\lambda,\tau,\alpha) = \left[1+\beta\left[\frac{e^{\{-\tau x^{\alpha}\}}}{1-e^{\{-\tau x^{\alpha}\}}}\right]^{\lambda}\right]^{-\theta} \tag{3.6}$$

The corresponding pdf of Equation (3.6), the h(x), H(x), and r(x) are given by

$$j_{IL-EW}(x;\theta,\beta,\lambda,\tau,\alpha) = \frac{\theta\beta\lambda\tau\alpha x^{\alpha-1}e^{\{-\lambda\tau x^{\alpha}\}}}{[1-e^{\{-\tau x^{\alpha}\}}]^{\lambda+1}}\left[1+\beta\left[\frac{e^{\{-\tau x^{\alpha}\}}}{1-e^{\{-\tau x^{\alpha}\}}}\right]^{\lambda}\right]^{-(\theta+1)} \tag{3.7}$$

$$v_{IL-EW}(x;\theta,\beta,\alpha,\lambda,\tau) = \frac{\theta\beta\lambda\tau\alpha x^{\alpha-1}e^{\{-\lambda\tau x^{\alpha}\}}}{\left[1-\left[1+\beta\left[\frac{e^{\{-\tau x^{\alpha}\}}}{1-e^{\{-\tau x^{\alpha}\}}}\right]^{\lambda}\right]^{-\theta}\right][1-e^{\{-\tau x^{\alpha}\}}]^{\lambda+1}}\left[1+\beta\left[\frac{e^{\{-\tau x^{\alpha}\}}}{1-e^{\{-\tau x^{\alpha}\}}}\right]^{\lambda}\right]^{-(\theta+1)} \tag{3.8}$$

$$K_{IL-EW}(x;\theta,\beta,\lambda,\tau,\alpha) = -log\left[1-\left[1+\beta\left[\frac{e^{\{-\tau x^{\alpha}\}}}{1-e^{\{-\tau x^{\alpha}\}}}\right]^{\lambda}\right]^{-\theta}\right] \tag{3.9}$$

$$r_{IL-EW}(x;\theta,\beta,\lambda,\tau,\alpha) = \frac{\theta\beta\lambda\tau\alpha x^{\alpha-1}e^{\{-\lambda\tau x^{\alpha}\}}}{\left[1+\beta\left[\frac{e^{\{-\tau x^{\alpha}\}}}{1-e^{\{-\tau x^{\alpha}\}}}\right]^{\lambda}\right][1-e^{\{-\tau x^{\alpha}\}}]^{\lambda+1}} \tag{3.10}$$

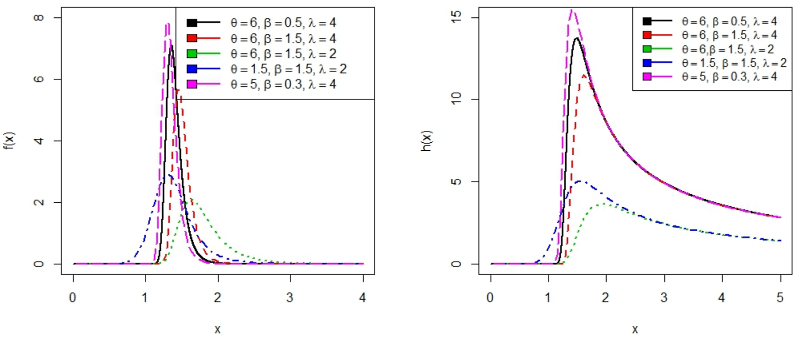

Fig. 2. Density and hazard rate plots of IL-W distribution with fixed $\alpha = 3.5$ and $\tau = 2$ and varying θ, β and λ.

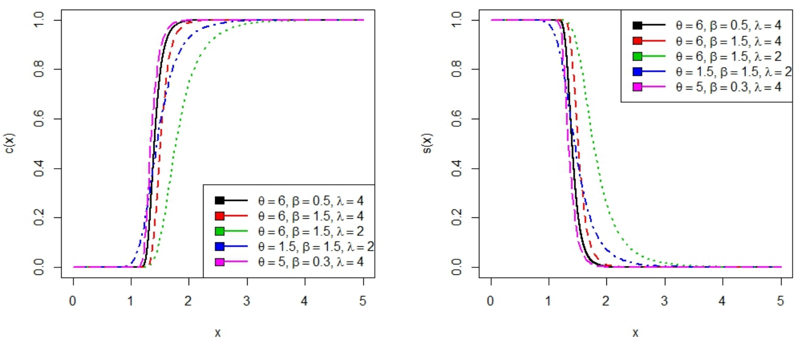

Fig. 3. cdf and survival function plots of IL-EBurr III distribution with fixed $\alpha = 3.5$ and $\tau = 2$ and varying θ, β and λ.

Fig. (2.) illustrates the various shapes of the density and hazard functions of the IL-EW based on some selected parameter values. The density can be skewed to the right and fairly symmetry (depending on parameters chosen). This include decreasing function. While Fig. 3. shows the cdf and survival functions of the IL-EW distribution.

3.3 The IL-EBIII Model

Lastly, if the parent distribution is Burr III, then

$$m(x;\tau,\alpha) = \alpha\tau x^{-(\alpha+1)}(1+x^{-\alpha})^{-(\tau+1)}$$

and

$$M(x;\tau,\alpha) = (1+x^{-\alpha})^{-\tau}$$

. With $x > 0$, $\tau, \alpha > 0$, then the IL-EBIII distribution has cdf given as:

$$J_{IL-EBIII}(x;\theta,\beta,\lambda,\tau,\alpha) = \left(1+\beta\left[\left(1+\frac{1}{x^{\alpha}}\right)^{\tau}-1\right]^{\lambda}\right)^{-\theta} \tag{3.11}$$

The corresponding pdf of Equation (3.11) is given as:

$$\begin{aligned} j_{IL-EBIII}(x;\theta,\beta,\lambda,\tau,\alpha) = \theta\beta\lambda\tau\alpha x^{-(\alpha+1)}(1+x^{-\alpha})^{(\lambda\tau-1)}\left[1-(1+x^{-\alpha})^{-\tau}\right]^{\lambda-1} \\ \left(1+\beta\left[(1+x^{-\alpha})^{(\tau)}-1\right]^{\lambda}\right)^{-(\theta+1)} \end{aligned} \tag{3.12}$$

The h(x), H(x), and r(x) are given by

$$\begin{aligned} v_{IL-EBIII}(x;\theta,\beta,\lambda,\tau,\alpha) = \frac{\theta\beta\lambda\tau\alpha x^{-(1+\alpha)}(1+\frac{1}{x^{\alpha}})^{(\lambda\tau-1)}\left[1-(1+x^{-\alpha})^{-\tau}\right]^{\lambda-1}}{\left[1-\left(1+\beta\left[(1+x^{-\alpha})^{(\tau)}-1\right]^{\lambda}\right)^{-\theta}\right]} \\ \times\left(1+\beta\left[(1+x^{-\alpha})^{(\tau)}-1\right]^{\lambda}\right)^{-(\theta+1)} \end{aligned} \tag{3.13}$$

$$K_{IL-EBIII}(x;\theta,\beta,\lambda,\tau,\alpha) = -log\left[1-\left(1+\beta\left[(1+\frac{1}{x^{\alpha}})^{(\tau)}-1\right]^{\lambda}\right)^{-\theta}\right] \tag{3.14}$$

$$r_{IL-EBIII}(x;\theta,\beta,\lambda,\tau,\alpha) = \frac{\theta\beta\lambda\tau\alpha x^{-(\alpha+1)}(1+x^{-\alpha})^{(\lambda\tau-1)}\left[1-(1+x^{-\alpha})^{-\tau}\right]^{\lambda-1}}{\left(1+\beta\left[(1+x^{-\alpha})^{(\tau)}-1\right]^{\lambda}\right)} \tag{3.15}$$

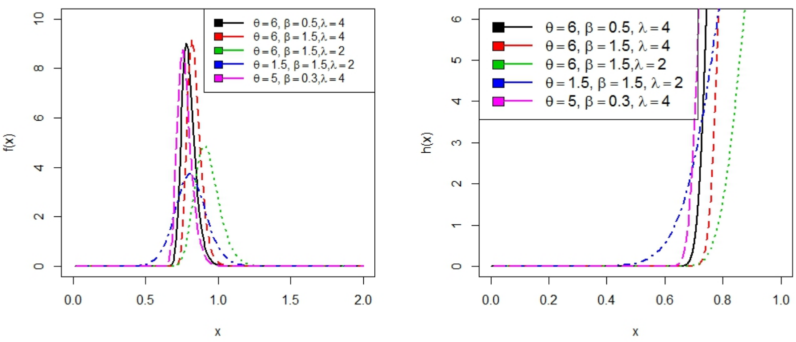

Fig. 4. Density and hazard rate plots of IL-EB III distribution with fixed $\alpha = 1.5$ and $\tau = 2$ and varying θ, β and λ.

Fig. (4.) illustrates the different shapes of the density and hazard functions of the IL-W at various parameter values.

4 Mixture Representations

In this section, we present the power series expansion of the IL-EG family by expanding Eqt. 2.1. Using binomial expansion

$$(1+y)^{-t} = \sum_{c=0}^{\infty} \binom{-t}{c} y^{-t-c}$$

(mathworld.wolfram.com/BinomialCoefficient.html)

$$J(x;\zeta) = \sum_{i=0}^{\infty} \binom{-\theta}{i} \beta^{-(\theta+i)} \left[\frac{\bar{M}(x;\upsilon)}{M(x;\upsilon)}\right]^{-\lambda(\theta+i)} \tag{4.1}$$

$$J(x;\zeta) = \sum_{i=0}^{\infty} \binom{-\theta}{i} \beta^{-(\theta+i)} \bar{M}^{-\lambda(\theta+i)}(x;\upsilon) G^{\lambda(\theta+i)}(x;\upsilon) \tag{4.2}$$

since

$$\bar{M}(x;\upsilon)^{-\lambda(\theta+i)} = \sum_{b=0}^{\infty} \frac{\Gamma\left(\lambda(\theta+b+i)\right)}{b!\Gamma\left(\lambda(\theta+i)\right)} M^{b}(x;\upsilon)$$

Then, Eqt. (4.2) can be written as

$$J(x;\zeta) = \sum_{i,b=0}^{\infty} \binom{-\theta}{i} \frac{\Gamma\left(\lambda(\theta+b+i)\right)}{b!\Gamma\left(\lambda(\theta+i)\right)} \beta^{-(\theta+i)} M^{\lambda(\theta+i)+b}(x;\upsilon) \tag{4.3}$$

Finally, Eqt. (4.3) can be written as

$$J(x;\zeta) = \sum_{i,b=0}^{\infty} \omega_{(i,b)} \Lambda_{(i,b)}(x;\upsilon) \tag{4.4}$$

where $\omega_{(i,b)} = \begin{pmatrix} -\theta \\ i \end{pmatrix} \frac{\Gamma(\lambda(\theta+b+i))}{b!\Gamma(\lambda(\theta+i))} \beta^{-(\theta+i)} \beta^{-(\theta+i)}$ and $\Lambda_{(i,b)}(x;\upsilon)$ is the cdf of the exponentiated-G family with power parameter $(\lambda(\theta+i)+b)$. The corresponding IL-Exponentiated G pdf is given by

$$j(x;\zeta) = \sum_{i,b=0}^{\infty} \omega_{(i,b)} \delta_{(i,b)}(x;\upsilon) \tag{4.5}$$

where $\delta_{(i,b)}(x;\upsilon) = [\lambda(\theta+i)+b]\, m(x;\upsilon) M^{[\lambda(\theta+i)+b-1)]}(x;\upsilon)$.

5 Mathematical Properties of IL-EG family

Here, we derived some of the mathematical properties of the IL-Exponentiated G family.

5.1 Moments

Suppose X denotes IL-Exponentiated G random variable with parameter space ζ, then the p^{th} moment about the origin is given by

$$E(X^p) = \int_0^{\infty} x^p j(x) dx = \int_0^{\infty} x^p \sum_{i,b=0}^{\infty} \omega_{(i,b)} \delta_{(i,b)}(x;\upsilon) dx, \qquad p = 0, 1, 2 \ldots \tag{5.1}$$

$$= \sum_{i,b=0}^{\infty} \omega_{(i,b)} \int_0^{\infty} x^p \left[\lambda(\theta+i)+b\right] m(x;\upsilon) M^{[\lambda(\theta+i)+b-1)]}(x;\upsilon) dx \tag{5.2}$$

$$= \sum_{i,b=0}^{\infty} \omega_{(i,b)} E(Z^p_{(i,b)}) \tag{5.3}$$

where $Z^p_{(i,b)}$ denotes the power-parameter Exp-G distribution $\lambda(\alpha+i+b)-1$.

5.2 Stress strength reliability

The reliability of stress strength is the likelihood of the part performing without fail, a defining feature for a given stress level under specified conditions. The reliability of stress strength (R_{IL-EG}) is given as

$$R_{IL-EG} = 1 - \int_{-\infty}^{\infty} \left[\sum_{i,b=0}^{\infty} \omega_{(i,b)} \delta_{(i,b)}(x;\upsilon) - \sum_{i,b=0}^{\infty} \omega_{(i,b)} \delta_{(i,b)}(x;\upsilon) \sum_{i,b=0}^{\infty} \omega_{(i,b)} \Lambda_{(i,b)}(x;\upsilon) \right] dx \tag{5.4}$$

5.3 Moment generating function

we defined the moment generating function (mgf) of IL-Exponentiated G as

$$M_X(t) = \int_{-\infty}^{\infty} \exp\{tx\} j(x) dx \tag{5.5}$$

By expanding Equation 5.5 using Taylor series,

$$M_X(t) = \sum_{p=0}^{\infty} \frac{t^p}{p!} \int_{-\infty}^{\infty} x^p j(x) dx \tag{5.6}$$

Substituting Equation (5.3) into the definition of $M_X(t)$ yields

$$M_X(t) = \sum_{p=0}^{\infty} \frac{t^p}{p!} E(X^p) \tag{5.7}$$

5.4 Lorenz and bonferroni curves

Lorenz and Bonferroni curves are inequality measures that have application in econometrics and insurance. Lorenz curve can be defined as $L_J(r) = \frac{\int_{-\infty}^{r} xj(x)dx}{\mu}$, so the Lorenz curve for IL-EG family can be expressed as

$$L_J(r) = \Pi_{i,b} \int_{-\infty}^{r} xm(x;\upsilon)M^{[\lambda(\theta+i)+b-1)]}(x;\upsilon) \tag{5.8}$$

where

$$\Pi_{i,b} = \frac{\sum_{i,b=0}^{\infty} \omega_{(i,b)} \left[\lambda(\theta+i)+b\right]}{\mu}$$

and the Bonferroni curve for IL-EG family is obtained as

$$B_J(r) = \Delta_{i,b} \int_{-\infty}^{r} xm(x;\upsilon)M^{[\lambda(\theta+i)+b-1)]}(x;\upsilon) \tag{5.9}$$

where

$$\Delta_{i,b} = \frac{\Pi_{i,b}}{\mu}$$

5.5 Order statistics

Order statistics are used across other areas of statistical theory and procedures, for instance, in identifying outliers across statistical quality control systems. Here, we extract the expressions in closed form for the pdf of the p^{th} order statistic of the IL-Exponentiated G family of distributions. Suppose $X_1, X_2, X_3, X_4 \ldots X_n$ are random samples from a distribution with pdf $j(x)$ and let $X_{1:n}, X_{2:n}, X_{3:n}, X_{4:n} \ldots X_{n:n}$ denotes the corresponding order statistics from this sample of size n, then

$$j_{p:n}(x;\vartheta) = \frac{n!j(x)}{(p-1)!(n-p)!} J(x)^{p-1} \left[1-J(x)\right]^{n-p} \tag{5.10}$$

where $j(x)$ and $J(x)$ are the pdf and CDF of the IL-EG distribution as in Eqt. (2.2) and Eqt. (2.1) respectively. By utilizing the fact that

$$\left[1-J(x)\right]^{n-p} = \sum_{k=0}^{n-p} (-1)^k \binom{n-p}{k} J(x)^k \tag{5.11}$$

By substituting (5.11) in 5.10 we have

$$f_{p:n}(x;\zeta) = \frac{n!j(x)}{(p-1)!(n-p)!} \sum_{k=0}^{n-p} (-1)^k \binom{n-p}{k} J(x)^{k+p-1} \tag{5.12}$$

also, by substituting $J(.)$ and $j(.)$ as in Eqt. (2.1) and Eqt. (2.2), Eqt. 5.12 becomes

$$j_{p:n}(x;\zeta) = \frac{n!\theta\lambda\beta m(x;\upsilon)\bar{M}^{\lambda-1}(x;\upsilon)}{M^{\lambda+1}(x;\upsilon)(p-1)!(n-p)!} \sum_{k=0}^{n-p} (-1)^k \binom{n-p}{k} \left(1+\beta\left[\frac{\bar{M}(x;\upsilon)}{M(x;\upsilon)}\right]^{\lambda}\right)^{-[\theta(k+p)+1]} \tag{5.13}$$

By enhancing the last term of Eqt. (5.13) and making some simplifications, we have

$$j_{p:n}(x;\zeta) = \sum_{k=0}^{n-p} \sum_{l,e=0}^{\infty} \sigma_{(kle)} m(x;\upsilon) M(x;\upsilon)^{\lambda\theta(k+p)+l+e-1} \tag{5.14}$$

where

$$\sigma_{(kle)} = \theta\lambda\beta^{-[\theta(k+p)+l]}(-1)^k \begin{pmatrix} n-p \\ k \end{pmatrix} \begin{pmatrix} -[\theta(k+p)+1] \\ l \end{pmatrix} \times \frac{n!\Gamma(\lambda\theta(k+p)+l+e+1)}{(p-1)!(n-p)!e!\Gamma(\lambda\theta(k+p)+l+1)} \tag{5.15}$$

and

$m(.)$ and $M(.)$ are the baseline pdf and cdf respectively.

5.6 Entropy

In this subsection, we consider the Renyi entropy by [27] and Shannon entropy by [28]. One measure of unknown variance is the entropy of a random variable X. The Renyi entropy for IL-Exponentiated G random variable is

$$I_R(\tau) = \frac{1}{(1-\tau)} log \left[\int_0^\infty j^\tau(x)dx \right], \tau > 0 \, and \, \tau \neq 1 \tag{5.16}$$

where from Eqt. (2.2)

$$j^\tau(x) = \left[\frac{\theta\beta\lambda m(x;\upsilon)\bar{M}^{\lambda-1}(x;\upsilon)}{M^{\lambda+1}(x;\upsilon)} \left(1+\beta\left[\frac{\bar{M}(x;\upsilon)}{M(x;\upsilon)}\right]^\lambda\right)^{-(\theta+1)} \right]^\tau$$

$$= \frac{(\theta\beta\lambda)^\tau m^\tau(x;\upsilon)\bar{M}^{\tau(\lambda-1)}(x;\upsilon)}{M^{\tau(\lambda+1)}(x;\upsilon)} \left(1+\beta\left[\frac{\bar{M}(x;\upsilon)}{M(x;\upsilon)}\right]^\lambda\right)^{-\tau(\theta+1)} \tag{5.17}$$

Eqt. (5.17) can be

$$j^\tau(x) = \sum_{i=0}^\infty \begin{pmatrix} -\tau(\theta+1) \\ i \end{pmatrix} \beta^{-(\tau\theta+i)} \frac{(\theta\lambda)^\tau m^\tau(x;\upsilon)\bar{M}^{\tau(\lambda-1)}(x;\upsilon)}{M^{\tau(\lambda+1)}(x;\upsilon)} \left[\frac{\bar{M}(x;\upsilon)}{M(x;\upsilon)}\right]^{-[\lambda\tau(\theta+1)+i]} \tag{5.18}$$

Then, by expanding $f^\tau(x)$ using a similar process as in Sec. (4) and some simplifications, yields

$$I_R(\tau) = \frac{1}{(1-\tau)} log \left[\sum_{j,i=0}^\infty \nu_{(j,i)} \int_0^\infty m^\tau(x;\upsilon) M^{[\tau(\lambda\theta-1)+i+j]}(x;\upsilon)dx \right] \tag{5.19}$$

where $\nu_{(i,j)} = (\theta\lambda)^\tau \beta^{-(\tau\theta+i)} \begin{pmatrix} -\tau(1+\theta) \\ i \end{pmatrix} \frac{\Gamma(\tau(\lambda\theta+1)+i+j)}{j!\Gamma(\tau(\lambda\theta+1)+i)}$.

Shannon Entropy is a Unique Case of Renyi entropy when $\tau \uparrow 1$ given by

$$E\{-log[j(x;\zeta)]\} = -log(\theta\beta\lambda) + E\left[-log\left[\frac{m(x;\upsilon)\bar{M}^{\lambda-1}(x;\upsilon)}{M^{\lambda+1}(x;\upsilon)}\right]\right] + (1+\theta)E\left[log\left(1+\beta\left[\frac{\bar{M}(x;\upsilon)}{M(x;\upsilon)}\right]^\lambda\right)\right] \tag{5.20}$$

6 Estimation

In this section, we present the maximum likelihood estimates (MLEs) of the parameters of the IL-Exponentiated G distribution. Let $x_1, x_2, x_3, \ldots, x_n$ be the observed values of n observations which are independently drawn from the IL-Exponentiated G distribution with parameter vector $\vec{\zeta} = (\theta, \beta, \lambda, \upsilon)^T$. The log-likelihood function for ζ denoted by $l(\zeta)$ can be written as

$$\begin{aligned} l(\zeta) &= nlog(\theta\beta\lambda) + \sum_{i=1}^{n} log\left(m(x_i;\upsilon)\right) + (\lambda-1)\sum_{i=1}^{n} log\left(\bar{M}(x_i;\upsilon)\right) \\ &-(\lambda+1)\sum_{i=1}^{n} log\left(M(x_i;\upsilon)\right) - (\theta+1)\sum_{i=1}^{n} log\left(1+\beta W(x_i;\upsilon)^{\lambda}\right) \end{aligned} \tag{6.1}$$

Where $W(x_i;\upsilon) = \left[\frac{\bar{M}(x_i;\upsilon)}{M(x_i;\upsilon)}\right]$. By using the partial derivatives of Eqt. (6.1) with respect to θ, β, λ, and υ, we derived the components of the score vector $U(\vec{\zeta})$ as follows

$$U_\theta(\zeta) = \frac{n}{\theta} - \sum_{i=1}^{n} log\left(1+\beta W(x_i;\upsilon)^{\lambda}\right) \tag{6.2}$$

$$U_\beta(\zeta) = \frac{n}{\beta} - \sum_{i=1}^{n} \frac{(\theta+1)W(x_i;\upsilon)^{\lambda}}{[1+\beta W(x_i;\upsilon)^{\lambda}]} \tag{6.3}$$

$$\begin{aligned} U_\lambda(\zeta) &= \frac{n}{\lambda} + \sum_{i=1}^{n} log\left(\bar{M}(x_i;\upsilon)\right) - \sum_{i=1}^{n} log\left(M(x_i;\upsilon)\right) \\ &-(\theta+1)\beta\sum_{i=1}^{n} \frac{W(x_i;\upsilon)^{\lambda} log\left(W(x_i;\upsilon)\right)}{[1+\beta W(x_i;\upsilon)^{\lambda}]} \end{aligned} \tag{6.4}$$

$$\begin{aligned} U_\upsilon(\zeta) &= \sum_{i=1}^{n} \frac{m'(x_i;\upsilon)}{m(x_i;\upsilon)} + (\lambda-1)\sum_{i=1}^{n} \frac{\bar{M}'(x_i;\upsilon)}{\bar{M}(x_i;\upsilon)} - (\lambda+1)\sum_{i=1}^{n} \frac{M'(x_i;\upsilon)}{M(x_i;\upsilon)} \\ &-\lambda\beta(\theta+1)\sum_{i=1}^{n} \frac{W(x_i;\upsilon)^{\lambda-1}W'(x_i;\upsilon)}{[1+\beta W(x_i;\upsilon)^{\lambda}]} \end{aligned} \tag{6.5}$$

Setting Equations (6.2, 6.3, 6.4, and 6.5) to zero and also solving simultaneously yields the MLE $(\hat{\zeta}) = (\hat{\theta}, \hat{\beta}, \hat{\lambda}, \hat{\upsilon})$ of ζ. However, these equations can't be solved analytically. Therefore, statistical software Can be used to find the maximum likelihood estimates of the parameters by using iterative methods.

7 Monte Carlo Simulation

In this section, a Monte Carlo simulation study is conducted and the results are presented to show the estimates' performance at various true parameter values. The study is divided into three sets as follows:

Set I, Set II, Set III have true parameter values (α= 0.5,β=0.8,λ=0.6,θ=0.5,τ=0.3),

(α=0.5,β=0.8,λ=1,θ=0.5,τ=0.3), and (α=0.5, β=0.8, λ=1.5,θ=0.5,τ=0.3), respectively.

The numerical study is described as follows:

(a). For true parameter values i.e $\zeta = (\theta, \beta, \lambda, \tau, \alpha)^T$, we simulated a random sample of size n from the IL-EBIII distribution using the quantile function defined in Equation (7.2).

(b). We then estimate the parameters of the IL-EBIII distribution from the sample using the method of maximum likelihood.

(c). We conduct N=1,000 replications of steps (a) and (b).

(d). For each of the five (5) estimated parameters of the IL-EBIII, from the N replicates, we compute the mean estimate, Bias, and MSE. The statistics are given by

$$\hat{\zeta} = \frac{1}{N}\sum_{i=1}^{N}\hat{\zeta}_i, \qquad Bias(\hat{\zeta}) = \hat{\zeta} - \zeta, \qquad var(\hat{\zeta}) = \sum_{i=1}^{N}\frac{(\hat{\zeta}_i - \hat{\zeta})^2}{N} \qquad MSE(\hat{\zeta}) = var(\hat{\zeta}) + (Bias(\hat{\zeta}))^2 \tag{7.1}$$

where the vector of estimated parameters $\hat{\zeta}_i$is the maximum likelihood estimate for each iteration $(n = 30, 70, 150, 300, 500, 1,000)$.The qf for IL-EBIII is giving by

$$Q_{IL-EBIII}(u) = \left[\left\{1 + \left[\frac{U^{\frac{-1}{\theta}} - 1}{\beta}\right]^{\frac{1}{\lambda}}\right\}^{\frac{1}{\tau}} - 1\right]^{-\frac{1}{\alpha}} \tag{7.2}$$

The simulation results are presented in Tables 1, 2, and 3, respectively. The simulation study has shown that irrespective of the parameter values chosen, the Bias and MSE of the parameter estimate for all the three sets decay as the sample size n increases. Thus, the larger the sample size, the more consistent are the estimates of the parameters. The estimates are good as they approach the true parameter values as the sample size increases.

Table 1. The Estimate, Bias, and MSE for set I(α= 0.5,β= 0.8,λ= 0.6,θ= 0.5,τ= 0.3)

n	Properties	α= 0.5	β= 0.8	λ= 0.6	θ= 0.5	τ= 0.3
30	Est.	1.1717	2.03398	0.7896	0.7288	0.7435
	Bias	0.6717	1.2398	0.1896	0.2288	0.4435
	MSE	1.574	6.6037	0.8519	0.8408	1.2019
70	Est.	0.9113	1.6316	0.7113	0.6565	0.6488
	Bias	0.4113	0.8316	0.1113	0.1565	0.3488
	MSE	0.7766	3.4006	0.4658	0.6116	0.6556
150	Est.	0.7792	1.3905	0.6737	0.5881	0.612
	Bias	0.2792	0.5905	0.0737	0.0881	0.312
	MSE	0.4255	1.8535	0.2989	0.4166	0.4855
300	Est.	0.7105	1.1838	0.6233	0.5419	0.5239
	Bias	0.2105	0.3838	0.0233	0.0419	0.2239
	MSE	0.2709	0.8453	0.1618	0.2858	0.2566
500	Est.	0.6347	1.0645	0.6084	0.533	0.4594
	Bias	0.1347	0.2645	0.0084	0.033	0.1594
	MSE	0.1261	0.4971	0.1061	0.1932	0.1548
1000	Est.	0.5911	0.9569	0.5758	0.5042	0.394
	Bias	0.0911	0.1569	-0.0242	0.0042	0.094
	MSE	0.0622	0.2008	0.0332	0.1012	0.0599

Table 2. The Estimate, Biase, and MSE for set II(α= 0.5,β= 0.8,λ= 1,θ= 0.5,τ= 0.3)

n	Properties	α= 0.5	β= 0.8	λ=1	θ=0.5	τ=0.3
30	Est.	1.4412	1.6245	1.1759	0.6782	0.8079
	Bias	0.9412	0.8245	0.1759	0.1782	0.5079
	MSE	2.6513	4.2383	1.1416	1.0273	1.3354
70	Est.	1.0265	1.2616	1.0172	0.6585	0.8696
	Bias	0.5265	0.4616	0.0171	0.1585	0.5696
	MSE	1.1122	1.9054	0.6213	0.9150	1.3311
150	Est.	0.8979	1.1122	0.9883	0.5755	0.8403
	Bias	0.3979	0.3122	-0.0117	0.0755	0.5403
	MSE	0.7192	0.9267	0.5258	0.6176	1.0913
300	Est.	0.8283	0.9591	0.9589	0.5815	0.7008
	Bias	0.3283	0.1591	-0.0411	0.0815	0.4008
	MSE	0.5836	0.4116	0.3353	0.4841	0.6379
500	Est.	0.7203	0.8637	0.9730	0.4838	0.6947
	Bias	0.2203	0.0637	-0.0269	-0.0162	0.3947
	MSE	0.2989	0.2178	0.2959	0.3073	0.4889
1000	Est.	0.6586	0.7966	0.9528	0.4301	0.6474
	Bias	0.1586	-0.0034	-0.0472	-0.0699	0.3474
	MSE	0.1828	0.0702	0.1833	0.1737	0.3409

Table 3. The Estimate, Bias, and MSE for set III(α= 0.5,β= 0.8,λ= 1.5,θ= 0.5,τ= 0.3)

n	Properties	α= 0.5	β= 0.8	λ= 1.5	θ= 0.5	τ= 0.3
30	Est.	1.4391	1.2569	1.6937	0.9583	0.5010
	Bias	0.9391	0.4569	0.1937	0.4583	0.2010
	MSE	2.508	2.7242	1.6903	1.8852	0.5577
70	Est.	1.1474	1.1585	1.4818	0.7300	0.5234
	Bias	0.6474	0.3585	-0.0182	0.2300	0.2234
	MSE	1.6526	2.0107	0.9568	0.7806	0.5018
150	Est.	0.8332	1.0675	1.5153	0.6482	0.5117
	Bias	0.3332	0.2675	0.0153	0.1482	0.2117
	MSE	0.6868	1.2565	0.7752	0.5355	0.3264
300	Est.	0.7721	0.9496	1.4428	0.5522	0.4899
	Bias	0.2721	0.1496	-0.0572	0.0522	0.1899
	MSE	0.5209	0.7258	0.5103	0.2491	0.2113
500	Est.	0.6156	0.9266	1.5059	0.5474	0.4655
	Bias	0.1156	0.1266	0.0059	0.0474	0.1655
	MSE	0.1738	0.4906	0.3469	0.1954	0.1748
1000	Est.	0.5415	0.8809	1.5645	0.4952	0.4283
	Bias	0.0415	0.0809	0.0645	-0.0048	0.1283
	MSE	0.0796	0.3118	0.2350	0.0836	0.0935

8 Application

We illustrate the application of the IL-EBIII distribution to two data sets; the data set of 20 patients undergoing an analgesic injection which were given relaxation periods, as reported by [29] and [30], and the data on strength of 1.5cm glass fiber as in [31] and [32].

We used an Adequacy Model package by [33] in R by [34]. The goodness of fit analytical measures as highlighted in [33] was used in comparing the performances of the models. Smaller values of the measures indicate better model fit. The estimated density plots of the data sets are presented in Fig. (5). Note that the competing models are given in Table 4. For the two datasets considered in this paper, the analytical measures and the estimated density plots suggest that the new IL-EBurr III distribution outperforms its competitors.

Table 4. Competing Models with IL-EBIII distribution

Models	References
GGBIII	[35]
KUMBIII	[36]
EBIII	[37]
WBIII	[38]

Table 5. MLEs and Log-likelihoods for the Data Sets

		MLEs					-log likelihood
Data Sets	Models	α	β	λ	θ	τ	
	ILEBIII	**1.8535**	**1.5341**	**1.8968**	**1.4661**	**1.325**	**38.4202**
	GGBIII	0.0243	1.7414	1.1212	0.9092	1.4462	232.4185
	EB	1.9389	1.7305	1.6034	1.9733	1.2482	46.4434
Glass fibre Data	KBIII	1.5071	1.5899	1.7449	1.5068		59.7296
	WBIII	0.3137	0.936	0.8037	0.4431		114.9281
	BIII	1.967	1.9941				63.6613
	ILEBIII	**1.9879**	**1.5036**	**1.8778**	**1.4449**	**1.9033**	**16.8855**
	GGBIII	0.6807	1.5415	1.9878	1.9989	1.4101	19.2738
	EBIII	1.5956	1.8298	1.6029	1.8135	1.2657	21.8749
Relief times Data	KBIII	1.8729	1.8871	1.8786	1.889		20.8097
	WBIII	0.7507	1.9059	1.223	1.7901		20.1203
	BIII	1.3535	1.5242				32.9266

Table 6. Goodness of Fits Statistics for the Data Sets

Data Sets	Models	AIC	CAIC	BIC	HQIC
	ILEBIII	**86.8404**	**87.8931**	**97.5561**	**91.0549**
	GGBIII	474.8369	475.8895	485.5526	479.0514
Glass fibre Data	KBIII	127.4591	128.1488	136.0317	130.8307
	WBIII	237.8561	238.5458	246.4286	241.2277
	BIII	131.3225	131.5225	135.6088	133.0083
	ILEBIII	**43.7709**	**48.0567**	**48.7497**	**44.7429**
	GGBIII	48.5476	52.8333	53.5263	49.5195
	EBIII	53.7499	58.0357	58.7287	54.7219
Relief times Data	KBIII	49.6194	52.2861	53.6023	50.3969
	WBIII	48.2406	50.9073	52.2236	49.0181
	BIII	69.8533	70.5591	71.8447	70.242

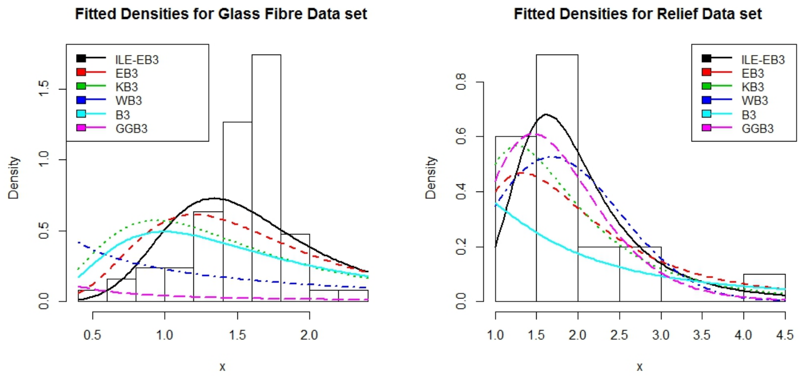

Fig. 5. Fitted Densities for the two data sets

9 Conclusion

In this paper, we proposed a new class of distributions called the Inverse Lomax-Exponentiated G (IL-EG) Family of Distributions. This family can extend several widely known models. For instance, we considered Weibull, Uniform, and Burr III as baseline distributions. We investigated some of its structural properties like an expansion for the density function using power series expansion. Some of the derived properties include Moments, Reliability, Moment generating functions, Inequality measures, quantile function, entropies, and order statistics. We estimated the parameters using the maximum likelihood method. The parameter estimates and the associated analytical measures showed that the new model based on the two data sets outperformed its competitors, thereby empirically showing the importance and value of the proposed family.

Competing Interests

Authors have declared that no competing interests exist.

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