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Conformal Vector Fields on Finsler Space with Special (α, β)- Metric

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this paper, we study the conformal vector fields on a class of Finsler metrics. In particular Finsler space with special (α, β)- metric $F = \alpha + \frac{\beta^2}{\alpha}$ is defined in Riemannian metric α and 1-form β and its norm. Then we characterize the PDE's of conformal vector fields on Finsler space with special (α, β)- metric.

Keywords: Finsler space; conformal vector fields; special (α, β)- metric.

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1 Introduction

The conformal theory of curves on Finsler geometry, emphasizing on the notion of circles preserving transformations, recently studied by the authors Z. Shen and Xia have studied conformal vector

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fields on a Randers manifold with certain curvature properties. Next, the conformal changes of metrics which leave invariant geodesic circles known as concircular transformation are characterized by a second order differential equation. The conformal vector fields are important in Riemann - Finsler geometry. Let $(M.F)$ be a Finsler manifold. It is known that a vector field $v = v^i \frac{\partial}{\partial x^i}$ on M is a conformal vector field on F with conformal factor $c = c(x)$ if and only if $X_v(F^2) = 4cF^2$, where $X_v = v^i \frac{\partial}{\partial x^i} + y^i \frac{\partial v^j}{\partial x^i} \frac{\partial}{partialy^j}$ [1]. They also determine conformal vector fields on a locally projectively flat Randers manifold. Besides they use homothetic vectorfields (c = constant) on Randers manifolds to construct new Randers metrics of scalar flag curvature [2]. Randers metrics seem to be among the simplest non - trivial Finsler metrics with many investigation in Physics, Electron optics with a magnetic field, dissipative mechanics, irreversible thermodynamics etc.

In this paper, we shall study the conformal vector fields on Finsler space with special (α, β)- metric, whose metric is defined in Riemannian metric α and 1-form β and its norm. Then we characterize the PDE's of conformal vector fields on Finsler space with special (α, β)- metric.
In natural way, we consider the general (α, β) - metrics are defined as the form:

$$F = \alpha\phi(b^2, \frac{\beta}{\alpha}). \tag{1.1}$$

This kind of metrics is first discussed by Yu and Zhu [3], Many well-known Finsler metrics are general (α, β) - metrics. For example, the Randers metrics and the square metrics are defined by functions $\phi = \phi(b^2, s)$ in the following form:

$$\phi = \frac{\sqrt{1 - b^2 + s^2} + s}{1 - b^2}. \tag{1.2}$$

$$\phi = \frac{(\sqrt{1 - b^2 + s^2} + s)}{(1 - b^2)^2\sqrt{1 - b^2 + s^2}}. \tag{1.3}$$

Based on the some reviews, further we shall study the covariant derivatives of conformal vector field is directly proportional to Finsler Special (α, β)- metric.

2 Preliminaries

Let M be an n-dimensional differentiable manifold and TM be the tangent bundle. A Finsler metric on M is the function $F = F(x, y) : TM \longrightarrow R$ satisfying the following conditions:

1. $F(x, y)$ is a C^∞ function on $TM\backslash\{0\}$;
2. $F(x, y) \geq 0$ and $F(x, y) = 0 \rightarrow y = 0$;
3. $F(x, \lambda y) = \lambda F(x, y), \lambda > 0$;
4. the fundamental tensor $g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2(F^2)}{\partial y^i \partial y^j}$ is positively defined.

Let

$$C_{ijk} = \tfrac{1}{4}[F^2]_{y^i y^j y^k} = \tfrac{1}{2} \tfrac{\partial g_{ij}}{\partial y^k}.$$

Define symmetric trilinear form $C = C_{ijk} dx^i \bigotimes dx^j \bigotimes dx^k$ on $TM\backslash\{0\}$. We call C be the Cartan torison tensor.
Let F be a Finsler metric on an n - dimensional manifold M. The canonical geodesic $\sigma(t)$ of F is characterized by

$$\tfrac{d^2\sigma^i(t)}{dt^2} + 2G^i(\sigma(t), \dot{\sigma(t)}) = 0,$$

where G^i are . 014/the geodesic coefficients having the expression $G^i = \frac{1}{4} g^{ij}\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\}$ with $(g^{ij}) = (g_{ij})^{-1}$ and $\dot{\sigma} = \frac{d\sigma^i}{dt}\frac{\partial}{\partial x^i}$. A spray on M is a globally C^∞ vector field G on $TM\backslash\{0\}$ which is expressed in local coordinates as follows

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}.$$

Given geodesic coefficients G^i, we define the covariant derivatives of a vector field as $XX^i(t)\frac{\partial}{\partial x^i}$ along a curve c(t) by

$$D_i X(t) = \{X^{\dot{i}}(t) + X^j(t) N^i_j(c(t), \dot{c(t)})\}\frac{\partial}{\partial x^i}|c(t)\ ,$$

where $N^i_j = \frac{\partial G^i}{\partial y^j}, X^{\dot{i}}(t) = \frac{dX^i}{dt}$ and $\dot{c} = \frac{dc^i}{dt}\frac{\partial}{\partial x^i}$.
Let $F = \alpha + \frac{\beta^2}{\alpha}$ be a Finsler Special (α, β)- metric expressed in terms of a Riemannian metric α and a vector field V on M.
From equation (1.1), where $\phi = \phi(b^2, s)$ is a positive smooth function on $[0, b_0) \times (-b_0, b_0)$. It is required that

$$\phi - \phi_2 s > 0, \ \ \phi - \phi_2 s + (b^2 - s^2)\phi_{22} > 0, \tag{2.1}$$

for $b < b_0$,where $\phi_1 = \frac{\partial \phi}{\partial b^2}, \phi_2 = \frac{\partial \phi}{\partial s}\phi_{22} = \frac{\partial^2 \phi}{\partial s^2}, \alpha = \frac{\sqrt{1-b^2+s^2}}{1-b^2}, \beta = \frac{s}{1-b^2}$.
We write the function where $\phi = \phi(b^2, s)$ in the following Taylor expansion

$$\phi = r_0 + r_1 s + r_2 s^2 + o(s^3),$$

where

$$r_i = r_i(b^2), \text{ and } r_0 = \frac{1}{(1-b^2)^{\frac{1}{2}}}, \ \ r_1 = \frac{1}{1-b^2}, r_2 = \frac{1}{2(1-b^2)^{3/2}}.$$

Now (1.2) implies that

$$r_0 > 0, \ \ r_0 + 2b^2 r_2 > 0.$$

But there is no restriction on r_1 . If we assume that $r_1 \neq 0$, then F is not reversible.(since F is not a symmetric see [4]).
Now, the Finsler Special (α, β) metric is on the conformal vector field, then Finsler Special (α, β)-metric becomes

$$\phi(b^2, s) = \frac{s^2}{(1-b^2)\sqrt{1-b^2+s^2}}$$

and (1.2) and (1.3) satisfy

$$\frac{1}{2b^2} + \frac{r_1^1}{r_1} - \frac{r_0^1}{r_0} + \left\{\frac{r_2}{r_0}[2\frac{r_1^1}{r_1} - \frac{r_0^1}{r_0}] - \frac{r_2^1}{r_0}\right\} b^2 = \frac{1}{2b^2(1-b^2)}. \tag{2.2}$$

Definition 2.1. Let F be a Finsler metric on a manifold M, and V be a vector field on M. Let ϕ_t be the flow generated by V. Define $\tilde{\phi} : TM \to TM$ by $\phi_t(x, y) = (\phi_t(x), \phi_t * (y))$. Then the vector field V is said to be conformal if

$$\phi_t^* F \tilde{=} e^{-2\sigma_t} F, \tag{2.3}$$

where σ_t is a function on M for every t. Differentiating the above equation by t at $t = 0$, we obtain

$$X_v(F) = -2cF, \tag{2.4}$$

where c is called the conformal factor and X_v is covariant derivative of vector field X, it can be defined as

$$X_v = V^i \frac{\partial}{\partial x^i} + y^i \frac{\partial V^j}{\partial x^i}\frac{\partial}{\partial y^j}, c = \frac{d}{dt}|_t = 0\sigma_t. \tag{2.5}$$

3 Conformal Vector Fields on Finsler Space with Special (α, β)- metric

In this section we shall study the conformal vector field on Kropina metric with (1.2). Let V be a conformal vector field of F with conformal factor c(x).

$$i.e., X_v(F^2) = 4cF^2. \tag{3.1}$$

Now we are in the position from (1.2) and to solve the above with the Kropina metric, we have

$$F = \alpha + \frac{\beta^2}{\alpha} = \frac{s^2}{(1-b^2)\sqrt{1-b^2+s^2}},$$

then (3.1) implies

$$X_v(F^2) = \phi^2 X_v(\alpha^2) + \alpha^2 X_v(\phi^2),$$

$$X_v(F^2) = [A_1 X_v(\alpha^2) + 2\alpha^2 X_v(b^2) A_2 + 2\alpha X_v(\beta) A_3 - 2\beta X_v(\alpha) A_4], \tag{3.2}$$

where,

$$A_1 = \tfrac{s^4}{(1-b^2)\sqrt{1-b^2+s^2}}, \quad A_2 = \tfrac{s^4}{(1-b^2)^3(1-b^2+s^2)^2},$$
$$A_3 = 0, \quad A_4 = 0$$

$$X_v(\alpha^2) = 2V_{0;0}, \quad X_v(\beta) = (V^j b_{i;j} + b^j V_{j;i}) y^i.$$

Again equation (3.2) equivalent to

$$B_1 V_{0;0} + \alpha B_2 (V^j b_{i;j} + b^j V_{j;i}) y^i + s^2 \alpha^2 X_v(b^2) - 2B_3 c\alpha^2 = 0 \tag{3.3}$$

$$B_1 = \frac{s^2}{(1-b^2)\sqrt{1-b^2+s^2}}, \quad B_2 = 0,$$
$$B_3 = \frac{s^2}{2}(1-b^2)^2(1-b^2+s^2)^{\frac{3}{2}} - \left\{\frac{s^2}{(1-b^2)\sqrt{1-b^2+s^2}}\right\}. \tag{3.4}$$

To simplify the computation, we fixed point $x \in M$ and make a co-ordinate change such that

$$y = \frac{s}{\sqrt{b^2-s^2}}\overline{\alpha}, \quad \alpha = \frac{b}{b^2-s^2}\overline{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2-s^2}}\overline{\alpha}, \quad \overline{\alpha} = \sqrt{\sum_{a=2}^{n}(y^a)^2}.$$

Then we have

$$V_{0;0} = V_{1;1}\frac{s^2}{b^2-s^2}\overline{\alpha^2} + (\overline{V}_{1;0} + \overline{V}_{0;1})\frac{s}{\sqrt{b^2-s^2}}\overline{\alpha} + \overline{V}_{0;0}, \tag{3.5}$$

$$V^j b_i + b^j V_{j;i} y^i = (V^j b_{1;j} + b^j V_{j;1})\frac{s}{\sqrt{b^2-s^2}}\overline{\alpha} + (V^j \overline{b}_{0;j} + b^j \overline{V}_{j;0}), \tag{3.6}$$

where,

$$\overline{V}_{1;0} + \overline{V}_{0;1} = \sum_{a=2}^{n}(V_{1;p} + V_{p;1})y^p, \quad \overline{V}_{0;0} = \sum_{p,q=0}^{n} V_{p;q} y^p y^q, \tag{3.7}$$

$$V^j\overline{b}_{0;j} + b^j\overline{V}_{j;0} = \sum_{p=2}^{n}(V^j b_{p;j} + b^j V_{j;p})y^p.$$

From (3.5) and (3.6) in to (3.3), which yields

$$\begin{aligned} B_1\{V_{1;1}\frac{s^2}{b^2-s^2}\overline{\alpha^2} + (\overline{V}_{1;0} + \overline{V}_{0;1})\frac{s}{\sqrt{b^2-s^2}}\overline{\alpha} + \overline{V}_{0;0}\}\} \\ +B_2\frac{b}{\sqrt{b^2=s^2}}\overline{\alpha}\{(V^j b_{1;j} + b^j V_{j;1})\frac{s}{\sqrt{b^2-s^2}}\overline{\alpha} + (V^j\overline{b}_{0;j} \\ +b^j\overline{V}_{j;0})\} + [s^2 X_v(b^2) - 2B_3 c]\frac{b^2}{b^2-s^2}\overline{\alpha^2} = 0. \end{aligned} \quad (3.8)$$

Consider the polynomial

$$\phi = r_0 + r_1 s + r_2 s^2 + o(s^3)$$

with $r_i = r_i(b^2)$ then we have,

$$\phi_1 = r_0^1 + r_1^1 s + r_2^1 s^2 + o(s^2).$$

By letting $s = 0$ in (3.8) we get,

$$r_0\overline{V}_{0;0} + r_1(V^j\overline{b}_{0;j} + b^j\overline{V}_{j;0})\overline{\alpha} + \{r_0^1 X_v(b^2) - 2cr_0\}\overline{\alpha^2} = 0. \quad (3.9)$$

According to the irrationality of $\overline{\alpha}$, the (3.8) is equivalent to

$$r_1(V^j\overline{b}_{0;j} + b^j\overline{V}_{j;0}) = 0, \quad (3.10)$$

$$r_0(\overline{V}_{0;0} + r_0^1 X_v(b^2) - 2cr_0)\overline{\alpha^2} = 0. \quad (3.11)$$

Therefore, the equation (3.10) yields

$$(V^j\overline{b}_{0;j} + b^j\overline{V}_{j;0}) = 0,$$

$$V^j b_{r;j} + b^j\overline{V}_{j;r} = 0. \quad (3.12)$$

Now, from equation (3.11), we have,

$$V_{r;s} + V_{s;r} = -2\{\frac{r_0^1}{r_0}X_v(b^2) - 2c\}\delta_{rs}, \quad 2 \le r, s \le n. \quad (3.13)$$

Again irrationality of $\overline{\alpha}$ from (3.3) we get

$$B_1(\overline{V}_{1;0} + \overline{V}_{0;1})\frac{s}{\sqrt{b^2-s^2}}\overline{\alpha} = 0. \quad (3.14)$$

$$\begin{aligned} B_1\{V_{1;1}\frac{b^2}{b^2-s^2}\overline{\alpha^2} + \overline{V}_{0;0}\} + B_2\frac{bs}{b^2-s^2}\overline{\alpha^2}(V^j b_{1;j} + b^j V_{j;1}) \\ +\{s^2 X_v(b^2) - 2cB_2\}\frac{b^2}{b^2-s^2}\overline{\alpha^2} = 0. \end{aligned} \quad (3.15)$$

From (3.13) we get

$$\overline{V}_{1;0} + \overline{V}_{0;1} = 0.$$

Which is equivalent to

$$V_{1;r} + V_{r;1} = 0. \tag{3.16}$$

Solving (3.11) for $\overline{V}_{0;0}$ and plugging it in to (3.15) we have

$$2B_1 s^2\{V_{1;1}\frac{r_0^1}{r_0}(X_v(b^2) - 2c)\} - 2\{\frac{r_0^1}{r_0}X_v(b^2) - 2c\}B_1(b^2)$$
$$+B_2 sb(V^j b_{1;j} + b^j V_{j;1}) + B_3 b^2 X_v(b^2) - 2cb^2 = 0. \tag{3.17}$$

By Taylor series, expansion of $\phi(b^2, s)$ then plugging it in to (3.15) and by the coefficients of s we have.

$$br_1(V^j b_{1;j} + b^j V_{j;1}) + b^2 X_v(b^2)\frac{\partial r_1}{\partial b^2} - 2cb^2 r_1 = 0. \tag{3.18}$$

Then

$$V^j b_{1;j} + b^j V_{j;1} = -(\frac{r_1^1}{r_1}X_v(b^2) - 2c)b_i. \tag{3.19}$$

From equation (3.13) and (3.19) which yields,

$$V^j b_{i;j} + b^j V_{j;i} = -(\frac{r_1^1}{r_1}X_v(b^2) - 2c)b_i. \tag{3.20}$$

Substituting (3.19) in (3.18), we get,

$$B_1 s^2\{V_{1;j} + (\frac{p_1^1}{p_0}X_v(b^2) - 2c)\} - b^2 X_v(b^2)\{\frac{p_0^1}{p_0}B_1 - B_2 \qquad +B_3 s\frac{p_1^1}{p_1}\} = 0. \tag{3.21}$$

The coefficients of all powers of s must vanish in (3.21). In particular, the coefficients of s^2 vanishes, the above equation becomes,

$$V_{1;1} + \frac{p_0^1}{p_0}X_v(b^2) - 2cb = -b^2 X_v(b^2)R_0, \tag{3.22}$$

where

$$R_0 = [\frac{p_0^1}{p_0}\frac{p_2}{p_0} + \frac{p_0^1}{p_0} - 2\frac{p_1^1}{p_1}\frac{p_2}{p_0}].$$

By (3.13),(3.16) and (3.22), we have

$$V_{i;j} + V_{j;i} = 4cp_{ij} - 2X_v(b^2)\{\frac{p_0^1}{p_0}p_{ij} + R_0 b_i b_j\}. \tag{3.23}$$

It is equivalent to

$$v_{i;j} + v_{j;i} = 4c\alpha - 2X_v(b^2)\{\frac{p_0^1}{p_0}\alpha + R_0\beta\}. \tag{3.24}$$

On contracting (3.16) with b^i and b^j yields

$$V_{i;j}b^i b^j = 2cb^2 - b^2 X_v(b^2)\{\frac{p_0^1}{p_0} + R_0 b^2\}. \tag{3.25}$$

Which is equivalent to

$$V_{i;j}b^i b^j = 2c\beta^2 - b^2 X_v(b^2).$$

Contracting (3.20) with b^i and b^j yields

$$V_{i;j}b^ib^j = 2cb^2 - b^2X_v(b^2)\{\frac{1}{2b^2} + \frac{p_1^1}{p_1}\}. \tag{3.26}$$

Here,we used the fact that $X_v(b^2) = 2b_{i;k}b^iV^k$. Then comparing (3.22) with (3.23), it yields

$$X_v(b^2)\{R_1 - R_0b^2\} = 0, \tag{3.27}$$

where $R_1 = \frac{1}{2b^2} + \frac{p_1^1}{p_1} - \frac{p_0^1}{p_0}$.
Therefore (3.27) reduced to

$$X_v(b^2)\{R_1 + R_2b^2\} = 0. \tag{3.28}$$

Here, two cases arises :
Case 1: If

$$R_1 + R_2b^2 \neq 0, \tag{3.29}$$

where, $R_2 = \frac{p_0^1}{p_0}\frac{p_2^1}{p_0} + \frac{p_2^1}{p_0} - 2\frac{p_1^1}{p_1}\frac{p_2}{p_0}$.
It follows from (3.29) that $X_v(b^2) = 0$ and in (3.19) and we have

$$V_{i;j} + V_{j;i} = 4c\alpha, \quad V^jb_{i;j} + b^jV_{j;i} = 2c\beta. \tag{3.30}$$

Notice that if $X_v(b^2) = 0$ and (3.30) holds then V satisfies (3.2) and V is an conformal vector field. Therefore, we obtain

Theorem 3.1. *Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on an n-dimensional manifold M $(n \geq 3)$ and let $V = V^i(x)\frac{\partial}{\partial x^i}$ be a conformal vector field.Then V is a conformal vector field of F with conformal factor $c = c(x)$ iff $X_v(b^2) = 0$ and*

$$V_{i;j} + V_{j;i} = 4c\alpha, \quad V^jb_{i;j} + b^jV_{j;i} = 2c\beta. \tag{3.31}$$

Case 2: If

$$R_1 + R_2b^2 = 0. \tag{3.32}$$

In this case $X_v(b^2) \neq 0$. Then obviously, we have

$$V_{i;j} + V_{j;i} = 4\bar{c}\alpha - 2X_v(b^2)b^{-2}R_1b_ib_j, \tag{3.33}$$

$$V^jb_{i;j} + V_{j;i}b^j = 2\bar{c}\beta. \tag{3.34}$$

Since V is conformal vectror field and from above equation (**??**) reduced to

$$X_v(b^2)\{B_1b^{-1}[(b^2 - s^2)R_1^*] + B_2 - (\frac{1 - b^2 + s^2}{s(1 - b^2)})\frac{p_1^1}{p_1} = 0. \tag{3.35}$$

Then, we obtain the theorem;

Theorem 3.2. *Therefore it follows we obtain Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on an n-dimensional manifold M $(n \geq 3)$ and let $V = V^i(x)\frac{\partial}{\partial x^i}$ be a conformal vector field.Then ,V is a conformal vector field of F with conformal factor $c = c(x)$ iff*

$$V_{i;j} + V_{j;i} = 4\bar{c}\alpha - 2X_v(b^2)b^{-2}R_1b_ib_j, \tag{3.36}$$

$$V^jb_{i;j} + V_{j;i} = 2\bar{c}\beta, \tag{3.37}$$

$$X_v(b^2)\{B_1b^{-1}[(b^2 - s^2)R_1^*] + B_2 - (\frac{1 - b^2 + s^2}{s(1 - b^2)})\frac{p_1^1}{p_1} = 0, \tag{3.38}$$

where,

$$R_1 = (\frac{1}{2b^2} + \frac{p_1^1}{p_1} - \frac{p_0^1}{p_0}),$$

$$R_1^* = (\frac{p_1^1}{p_1} - \frac{p_0^1}{p_0})\frac{s^2}{2b^2},$$

$$B_1 = \frac{b^4(1+s^2) - 4b^2}{s(1-b^2)^2},$$

$$B_2 = \frac{\alpha^2 - b^4 - b^2(s^2-2) - 1}{s^2(1-b^2)^2},$$

$$\bar{c} = c - \frac{1}{2}X_v(b^2)\frac{p_0^1}{p_0}. \tag{3.39}$$

4 Conclusions

Conformal vector fields play an important role in Finsler geometry. When F is Riemannian mertic, the local solutions of a conformal vector field can be determined if F satisfies certain conditions. As we know every conformal vector field is associated with scalar function called conformal factor.

In this paper we study the conformal vector field on Finsler - Kropina metric and characterize conformal vector fields on Kropina metric in terms of PDE's.

Competing Interests

Authors have declared that no competing interests exist.

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