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Bounded Oscillation Theorem for Unstable-type Neutral Impulsive Differential Equations of the Second Order

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

The oscillations theory of neutral impulsive differential equations is gradually occupying a central place among the theories of oscillations of impulsive differential equations. This could be due to the fact that neutral impulsive differential equations plays fundamental and significant roles in the present drive to further develop information technology. Indeed, neutral differential equations appear in networks containing lossless transmission lines (as in high-speed computers where the lossless transmission lines are used to interconnect switching circuits). In this paper, we study the behaviour of solutions of a certain class of second-order linear neutral differential equations with impulsive constant jumps. This type of equation in practice is always known to have an unbounded non-oscillatory solution. We, therefore, seek sufficient conditions for which all bounded solutions are oscillatory and provide an example to demonstrate the applicability of the abstract result.

Keywords: Second-order; impulsive; neutral delay differential equation; oscillation.

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1 Introduction and Statement of the Problem

Consider the second order linear neutral impulsive differential equations of the form

$$\begin{cases} \left[y(t)-py(t-\tau)\right]'' = q(t)y(g(t)), \ t \ge t_0, \ t \notin S \\ \Delta\left[y(t_k)-py(t_k-\tau)\right]' = q_k y(g(t_k)), \ t_k \ge t_0, \ \forall\, t_k \in S, \end{cases} \tag{1.1}$$

where $0 \le t_0 < t_1 < \cdots < t_k < \cdots$ with $\lim_{k\to+\infty} t_k = +\infty$, $\Delta y^{(i)}(t_k) = y^{(i)}(t_k^+) - y^{(i)}(t_k^-), \ i = 0,1$ and $y(t_k^-), \ y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively. Except otherwise specified, throughout this discussion we shall assume the following conditions are satisfied without further mention:

i) $p \in R$, $\tau > 0$, $q_k \ge 0$,

ii) $q \in PC([t_0,\infty), R_+)$,

iii) $g \in C([t_0,\infty), R)$,

iv) $\lim_{t\to\infty} g(t) = \infty$.

Since Sturm's famous memoir in the 17th century, it is observed that a great deal of interest has been focused on the behaviour of solutions of ordinary and delay differential equations in spite of the existence of extensive literature in these fields [1,2,3-5,6]. Still more interesting, the theory of impulsive differential equations has brought in yet another dimension to the whole scenario and has helped to usher in a new body of knowledge for further considerations. The effects of these new inputs can be observed in the study of oscillatory properties of impulsive differential equations with deviating arguments as well as the investigation of neutral impulsive differential equations which have recently captured the attention of many applied mathematicians as well as other scientists around the world [7,8,9,10,11,12].

Neutral delay impulsive differential equations contain the derivative of the unknown function both with and without delays. Some new phenomena can appear, hence the theory of neutral delay impulsive differential equations is even more complicated than the theory of non-neutral delay impulsive equations. The oscillatory behavior of the solutions of neutral systems is of importance in both the theory and applications, such as the motion of radiating electrons, population growth, the spread of epidemics, in networks containing lossless transmission lines (see [1,13,14,15] and the references therein).

The above definition in equation (1.1) becomes more meaningful if we define other related terms and concepts that will continue to be useful as we progress through the article.

In ordinary differential equations, the solutions are continuously differentiable, sometimes at least once, whereas impulsive differential equations generally possess non-continuous solutions [16,17]. Since the continuity properties of the solutions play an important role in the analysis of the behaviour, the techniques used to handle the solutions of impulsive differential equations are fundamentally different, including the definitions of some of the basic terms. In this section, we examine some of these changes:

Let an evolution process evolve in a period of time J in an open set $\Omega \subset J \times R^n$ and let the function $f : \Omega \to R^n$ be at the least a continuous mapping fulfilling local Lipchitzian condition in $y \in R^n, \ \forall\, (t,y) \in \Omega$. Let the real numerical sequence $S = \{t_k\}_{k=1}^{\infty} \subset J$ be increasing without finite accumulation point such that $0 \le t_0 < t_1 < \cdots < t_k < \cdots$ with $\lim_{k \to +\infty} t_k = +\infty$. The points $t_k, k \in N$ are called moments of impulse effect. Then the governing second order impulsive differential equation is of the form

$$\begin{cases} y''(t) = f(t, y, y'), \ t \ne t_k \\ \Delta y'(t_k) = f_k(y, y'), \ t = t_k, \end{cases}$$

where $y' = \frac{dy}{dt}$, $y'' = \frac{d^2 y}{dt^2}$, $(t, y(t)) \in \Omega$, $\Delta y^{(i)}(t_k) = y^{(i)}(t_k^+) - y^{(i)}(t_k^-)$, $i = 0,1$ and $y(t_k^-), \ y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively. For the sake of definiteness, we shall suppose that the functions $y(t)$ and $y'(t)$ are continuous from the left at the points t_k such that $y'(t_k^-) = y'(t_k)$, $y(t_k^-) = y(t_k)$.

For the description of the continuous change of such processes, ordinary differential equations are used, while the moments and the magnitude of the change by jumps are given by the jump conditions. Now, in the case of unfixed moments of impulse effects, the impulse points may be time and state-dependent, that is, $t_k = t_k(t, y(t))$. When the function t_k depends on the state of the system (1.1), then it is said to have impulses at variable times. This is reflected in the fact that different solutions will tend to undergo impulses at different times.

In this paper, we shall restrict ourselves to the investigation of conditions for bounded oscillatory solutions of impulsive differential equations for which the impulse effects take place at fixed moments of time $\{t_k\}$. Our equation under consideration is of the form in equation (1.1), where $t, t_k \ge 0, \ k \in N$. Without further mentioning, we will assume throughout this paper that every solution $y(t)$ of equation (1.1) that is under consideration here, is continuous from the left and is nontrivial. That is, $y(t)$ is defined on some half-line $[T_y, \infty)$ and $\sup \{|y(t)| : \ t \ge T\} > 0$ for all $T \ge T_y$. Such a solution is called a regular solution and we are only interested in the behaviour of the regular solutions $y(t)$ of equation (1.1) and assume that the equation under consideration possesses such solutions.

We say that a real valued function $y(t)$ defined on an interval $[a, +\infty)$ fulfills some property *finally* if there exists $T \ge a$ such that $y(t)$ has this property on the interval $[T, +\infty)$.

Definition 1.1: The solution $y(t)$ of the impulsive differential equation (1.1) is said to be

i) Finally positive (finally negative) if there exist $T \ge 0$ such that $y(t)$ is defined and is strictly positive (negative) for $t \ge T$ [8];
ii) Non-oscillatory, if it is either finally positive or finally negative; and
iii) Oscillatory, if it is neither finally positive nor finally negative [7,9].

Remark 1.1: All functional inequalities considered in this paper are assumed to hold finally, that is they are satisfied for all t large enough.

Definition 1.2: We say that a real-valued function $y(t)$ is the solution of equation (1.1) if there exists a number $t_0 \in R$ such that $y(t) \in PC([t_0-\tau,\infty),R)$, the function $y(t)+p(t)y(t-\tau)$ is twice continuously differentiable for $t \geq t_0-\tau,\ t \neq t_k,\ k \in N$ and $y(t)$ satisfies equation (1.1) for all $t \geq t_0-\tau$.

The second order neutral impulsive differential equation (1.1) is a system consisting of a differential equation together with an impulsive condition in which the second order derivative of the unknown function appears in the equation both with and without delay. Second order neutral impulsive differential equations have numerous applications to problems arising in mechanics, electrical engineering, medicine, biology, ecology, etc, and in mathematical and theoretical physics. Its application also appears, for instance, in problems dealing with vibrating masses attached to an elastic. They also appear, as the Euler equation, in some vibrational problems [7,1]. This work is inspired by the fact that not much has been done in the area of qualitative behaviour of the solutions of second-order neutral delay impulsive differential equations. In general, equation (1.1) always has an unbounded non-oscillatory solution. Our aim in this study is to establish sufficient conditions for which all bounded solutions of equation (1.1) are oscillatory.

Remark 1.2: Without loss of generality, we will deal only with the positive solutions of equation (1.1).

2 Main Results

The following theorem is an extension of Theorem 4.6.3 found on page 258, being its neutral delay version as identified in the work by Erbe et al. [6].

Theorem 2.1: Assume that

i) $p=1,\ q_k>0\ \text{and}\ \tau>0;$
ii) $g(t) \leq t$ and g is non-decreasing for $t \geq t_0$;
iii) Either

$$\int_{t_0}^{\infty} t\,q(t)\,dt + \sum_{t_0 \leq t_k < \infty} t_k q_k = \infty \tag{2.1}$$

or

$$\lim_{t,t_k \to \infty} \left[t \int_{t}^{\infty} q(s)\,ds + t_k \sum_{t \leq t_k < \infty} q_k \right] = \infty. \tag{2.2}$$

Then every bounded solution of equation (1.1) is oscillatory.

Proof: Let us assume, by contradiction, that $y(t)$ is a bounded finally positive solution of equation (1.1). Define

$$z(t) = y(t) - p\,y(t-\tau). \tag{2.3}$$

Then equation (1.1) becomes

$$\begin{cases} z''(t)=q(t)y(g(t)), & t\notin S \\ \Delta z'(t_k)=q_k y(g(t_k)), & t_k\in S. \end{cases} \tag{2.4}$$

Since $g(t)$ is a non-decreasing function, we have that

$$\begin{cases} z''(t)=q(t)y(g(t))\geq 0, & t\notin S \\ \Delta z'(t_k)=q_k y(g(t_k))\geq 0, & t_k\in S, \end{cases}$$

which implies that $z'(t)$ is a non-decreasing function of t. From the above observation, it follows that either

$$\lim_{t\to\infty} z(t)=\lim_{t\to\infty} z'(t)=-\infty \tag{2.5}$$

or

$$\lim_{t\to\infty} z'(t)=-\ell<\infty \tag{2.6}$$

We clearly see that condition (2.5) implies $z'(t)\leq 0$ and $z(t)>0$ finally. Now, we assume that condition (2.6) holds. Integrating both sides of equation (2.4) from t_0 to t, and letting $t\to\infty$, we obtain

$$\int_{t_0}^{\infty} q(s)y(g(s))ds+\sum q_k y(g(t_k))=z'(t_0)-\ell$$
.

Hence,

$$\int_{t_0}^{\infty} q(s)y(g(s))ds<\infty, \quad \sum_{t_k\geq t_0}\Delta z'(t_k)<\infty,$$

which implies that $y(t)\in L_1[t_0,\infty)$ and so, from equation (2.3), $z(t)\in L_1[t_0,\infty)$ where $L_1[t_0,\infty)$ is the space of all Lebesgue integrable functions on the interval $[t_0,\infty)$. This follows that

$$\lim_{t\to\infty} z(t)=\lim_{t\to\infty} z'(t)=0 \tag{2.7}$$
,

and therefore $\ell=0$. Finally, by equations (2.7) and (2.6) with $\ell=0$ and the non-increasing nature of $z'(t)$, we have that $z'(t)\leq 0$ and $z(t)<0$ finally.

Here we see that there are two possibilities for $z(t)$:

a) $z''(t), \Delta z'(t_k)\geq 0, \quad z'(t), \Delta z(t_k)\leq 0, \quad z(t)<0$ for $t, t_k\geq t_1\geq t_0$;

b) $z''(t),\ \Delta t'(t_k)\geq 0,\quad z'(t),\ \Delta z(t_k)\leq 0,\quad z(t)>0$ for $t, t_k\geq t_1\geq t_0$.

In case (a), there exists a finite number $\alpha>0$ such that

$$\lim_{t\to\infty} z(t)=-\alpha.$$

Thus, there exists $t_2\geq t_1$ such that $-\alpha<z(t)<-\frac{\alpha}{2},\quad t\geq t_2,$ that is, $-\alpha<y(t)-y(t-\tau)<-\frac{\alpha}{2},\quad t\geq t_2$, where t_2 is a sufficiently large number. Using equation (2.3) and the fact that $p=1$, we have $y(t-\tau)>\frac{\alpha}{2},\ t\geq t_2.$ Since $\lim_{t\to\infty} g(t)=\infty$, then there exists $t_3\geq t_2$ such that $g(t)\geq t_2+\tau$ for $t\geq t_3\geq t_2$. Thus, we have

$$y(g(t))>\frac{\alpha}{2},\ t\geq t_3 \tag{2.8}$$

Combining equations (2.8) and (2.4), we have

$$\begin{cases} z''(t)\geq\frac{\alpha}{2}q(t),\ t\geq t_3,\ t\notin S \\ \Delta z'(t_k)\geq\frac{\alpha}{2}q_k,\ t_k\geq t_3, \forall\ t_k\in S. \end{cases} \tag{2.9}$$

In case (b), we have

$$y(t)>y(t-\tau),\ t\geq t_1.$$

Then there exists $L>0$ such that

$$y(t)\geq L,\ t\geq t_1. \tag{2.10}$$

Substituting equation (2.10) into equation (2.4), we have

$$\begin{cases} z''(t)\geq Lq(t),\ t\geq t_3,\ t\notin S \\ \Delta z'(t_k)\geq Lq_k,\ t_k\geq t_3,\ t_k\in S. \end{cases} \tag{2.11}$$

Therefore, in both cases, we are led to the same inequality (2.11). According to the discussion above, there always exists positive constants L_0 and $T^*>t_1$ such that

$$\begin{cases} z''(t)\geq L_0 q(t),\ t\geq T^*,\ t\notin S \\ \Delta z'(t_k)\geq L_0 q_k,\ t_k\geq T^*,\ t_k\in S. \end{cases} \tag{2.12}$$

Integrating inequality (2.12) from t to T, we have

$$z'(T)-z'(t)\geq L_0\left[\int_t^T q(s)ds+\sum_{t\leq t_k\leq T} q_k\right],\quad T^*\leq t,\, t_k<T.$$

Hence,

$$-z'(t)\geq L_0\left[\int_t^T q(s)ds+\sum_{t\leq t_k\leq T} q_k\right],\quad T^*\leq t,\, t_k<T,$$

which implies that

$$\int_{t_0}^{\infty} q(s)ds+\sum_{t_0\leq t_k<\infty} q_k<\infty,$$

and so

$$-z'(t)\geq L_0\left[\int_t^{\infty} q(s)+\sum_{t\leq t_k<\infty} q_k\right]. \tag{2.13}$$

Integrating inequality (2.13) from t to T for $T>t\geq T^*$ and letting $T\to\infty$, we obtain

$$z(t)\geq z(T)+L_0\left[\int_t^T\int_s^{\infty} q(u)duds+\sum_{t\leq t_k\leq T}\sum_{s\leq t_k<\infty} q_k\right]$$

$$=z(T)+L_0\left[\int_t^T (u-t)q(u)du+(T-t)\int_T^{\infty} q(u)du+\sum_{t\leq t_k\leq T}(t_k-t)q_k+(T-t)\sum_{T\leq t_k<\infty} q_k\right],\quad t,t_k\geq t_2,$$

which leads to a contradiction to the boundedness of $z(t)$ in either of the cases in equation (2.1) or (2.2). This completes the proof of Theorem 2.1.

Example 2.1: Consider the equation

$$\begin{cases}\left[y(t)-y(t-2\pi)\right]''=\dfrac{2\pi}{t-\pi}y(t-\pi)\\ \Delta y(t_k)-y(t_k-2\pi)\big]'=\dfrac{2\pi}{t_k-\pi}y(t_k-\pi).\end{cases} \tag{2.14}$$

It is easy to see that all the assumptions of Theorem 2.1 are satisfied. Therefore, every bounded solution of equation (2.14) is oscillatory. Equation (2.1) may have unbounded oscillatory solutions. For example, equation (2.14) has $y(t)=t\sin t$ as such a solution.

3 Conclusion

In this paper, we are mainly concerned with oscillating systems which remain oscillating after being perturbed by instantaneous changes of state or impulsive constant jumps. We considered a certain type of second-order neutral delay differential system and provided sufficient conditions governing the impulse operators acting on the system so that its bounded solutions are oscillatory. Roughly speaking, by the proper imposition of impulses in Theorem 2.1, the oscillatory properties of the solutions of the neutral differential equation referred to in Theorem 4.6.3 of [6] have been preserved. Here, we are able to demonstrate how well-known mathematical techniques and methods, after suitable modification, is extended in proving an oscillatory theorem for a class of second-order neutral impulsive differential equations (1.1). The salient techniques for the proof were obtained from studies by Bainov and Simeonov [7].

Competing Interests

Authors have declared that no competing interests exist.

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