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New Approach to Exponential H_∞ Filtering for Uncertain Singular Systems with Markovian Switchings

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this paper, the problem of exponential H_∞ filtering is investigated for uncertain singular Markovian jump systems(SMJSs) with time-varying delay. A mode dependent approach is presented to deal with it. By constructing a stochastic Lyapunov functional and employing some novel integral inequalities, a filtering design criterion is proposed under which the resulting filtering error system is robustly exponentially mean-square admissible with a prescribed H_∞ performance index. Based on this, the proper gain matrices and optimal H_∞ performance index can be efficiently obtained via solving a convex optimization problem subject to some linear matrix inequalties(LMIs). Finally, a numerical example is given to demonstrate the reduced conservatism and effectiveness of the presented general filtering technique.

Keywords: *H_∞ filtering; Markovian jump systems; exponentially mean-square addmissible; time-varying delay; singular systems.*

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1 Introduction

Singular systems, which are also referred to as generalized state-space systems, descriptor systems or implicit systems, as a kind of important system, singular systems have been received extensively attention during the past decade due basically to their powerful applications in many practical systems, such as economic systems, robotic systems, biological systems network control systems, chemical systems, and many other systems. Different from other regular systems, singular systems are more general and complex owing to not only the stability is need to be considered, but also regularity and absence of impulses or causality is need to be considered. Due to this fact that singular systems can better describe and analyze the behavior of some physical systems than regular ones standard state-space systems [1],[2]. Moreover, since the existence of time-delays often causes undesirable behavior such as degradation stability in dynamical systems. So one of the major issues of stability and control analysis for time-delay singular systems has been studied extensively in actual problem [3],[4].

As a fundamental issue in systems and control theory, filtering design problem is a very significant research topic in signal processing and systems control. In practice, as well known, some state variable of certain systems cannot be measured directly, in order to deal with this gap, the filtering has been considered as the most important approaches to estimate the state information when the plan is disturbed Currently, there are many approaches proposed for filtering design, such as Kalman filtering, stationary filtering, varance-constrained filtering, fusion filtering, fault detection filtering, L_1/L_2 filtering, $L_2 - L_\infty$ filtering[5], H_2/H_∞ filtering[6],[7], H_∞ filtering[8] and so on. As well known, among the existing filtering methods, the problem of H_∞ filtering is very important and challenging strategy. For the state variables estimation of dynamic systems, the H_∞ filtering is to design an estimate of the unknown state combination via output measurements, which minimizes H_∞ norm of the transfer function between external disturbance noise and estimation error. In contrast with the other filtering, the advantage of the H_∞ filtering is that the noise signals setting are arbitrary signals with bounded energy and does not assumption on the statistic of the process and measurement noise, which is more general than classical Kalman filtering. Recently, many results about H_∞ filtering problem have been reported[9],[10],[11],[12],[13],[14],[15],[16],[17], for example [11] the resilient finite-time H_∞ filtering problem for discrete-time uncertain Markov jump neural networks with dropouts is investigated; [12]concerned the H_∞ filtering problem for a class of networked systems subject to randomly occurring distributed state delays, nonlinearities, sensor saturation as well as missing measurements via unreliable communication channels ; [16] studied the probability-guaranteed H_∞ finite-horizon filtering problem, the system matrices are functions of mutually independent stochastic variable that obey uniform distributions over known finite ranges, which has not been considered in the existing references, is proposed.

It is well known that stochastic modelings have come to play important roles in many branches of science and engineering applications[18],[19],[20], and Markovian jump systems is one of the important stochastic hybrid systems which have received considerable research attentions. In practice, the structure and parameters of the dynamic systems may be changed abruptly due to the randomly occurring phenomena such as component failure and repairs of the components, change in the interconnections of subsystems and sudden environment changes and so on [21],[22],[23],[24]. Such an abrupt change could be molded by jump systems and, in the past decade, increased attention has been devoted to the control or filtering problem for Markovian jump systems. [25],[26],[27] considered the H_∞ filtering for Markovian jump systems with non-accessible mode information or partly unknown transition probability ; [28] deal with the robust H_∞ filtering for a class of uncertain discrete-time nonhomogeneous Markov jump systems and the time-varying jump transition probability matrix is described by a polytope; [29] addressed the problem of finite-time H_∞ filtering for a class of interconnected Markovian jump system with mode-dependent delays in the interconnection. To the best of the author's knowledge, however, most of above references about H_∞ filtering design problem only for the regular systems. In comparison with the regular systems, singular systems are more complex and difficulty, because it requires to consider regularity and impulse-free or causality, which do not arise in the regular ones, particularly, if the singular matrices and Markovian jump parameters are contained simultaneously, the synthesis of SMJSs will become more complicated[30],[31]. To the best our knowledge, some essential results on regular systems can not be extend to SMJSs directly or it is very difficulty to do this. So, there is still room for investigation of many difficult problem for singular systems.

Motivated by the aforementioned observations, in this paper, we will deal with the problem of H_∞ filtering design for a class of SMJSs with parameters uncertainties which based on some novel integral inequalities and improving methods. The purpose of this paper is to design mode-dependent singular filters such that the filtering error systems is not only robustly exponentially mean-square admissible but also satisfies a prescribed H_∞ norm level for all admissible uncertainties. A novel stochastic Lyapunov function and a set of strict LMIs will be utilized to derive sufficient conditons guaranteeing the desired filter can be constructed. Finally, a numerical example is proposed to illustrate the effectiveness of the obtained results.

The main contributions of this paper include the following: (1). a comprehensive stochastic Lyapunov function is constructed to reflect the information of Markovian jump modes and the inherent state time-varying delays; (2). the difficult problem of design filtering parameters are solved by utilizing several well-known matrix inequationsnonel integral inequations and convex optimization technique; (3). the desired H_∞ filter design method can be extended to cope with fault detection filtering and H_∞ deconvolution filtering with more generally mode transition information easily.

Notations: The notations are quite standard. Throughout this letter $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensioned Euclidean space and the set of all $n \times m$ real matrix. The notation $X \geq Y$. (respective$X > Y$) means that X and Y are symmetric matrices, and that $X - Y$ is positive semi-definitive (respective positive definite). $L_2\left[0, +\infty\right)$ is square integrable function vector over $[0, +\infty)$. $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$. I is the identity matrix with appropriate dimensions. If A is a matrix , $\lambda_{\max}\left(A\right)$ (respective $\lambda_{\min}\left(A\right)$) means the largest (respective smallest) eigervalue of A . Moreover, let $(\Omega, F, (F_t)_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filteration .$(\mathbb{F}_t)_{t\geq 0}$ satisfies the usual conditions (i.e, the filtration contains all P-null sets and is right continuous). $E\left\{\cdot\right\}$ stands for the mathematical expectation operator with respect to the given probability measure. Denote by $L^2_{\mathbb{F}_0}\left(\left[-d_2, 0\right] : \mathbb{R}^n\right)$ the family of all $\mathbb{F}_0$ measurable $C\left(\left[-d_2, 0\right] : \mathbb{R}^n\right)$-valued random variables $\varphi = \left\{\varphi\left(s\right) : -d_2 \leq s \leq 0\right\}$ such that $\sup_{-d_2\leq s\leq 0} E\left\|\varphi\left(s\right)\right\|^2 < \infty$. The asterisk * in a matrix is used to denote term that is induced by symmetry. Matrices, if not explicitly, specified, are assumed to have appropriate dimensions. Sometimes, the arguments of function will be omitted in the analysis when no confusion can be arised.

2 Problem Formulation and Preliminaries

Consider a class of SMJSs with time-varying delay and norm-bounded parametric uncertainties described by the following model:

$$\begin{aligned} E\left(r_t\right)\dot{x}\left(t\right) &= A\left(t, r_t\right)x\left(t\right) + A_d\left(t, r_t\right)x\left(t - d\left(t\right)\right) + B\left(r_t\right)\omega\left(t\right) \\ y\left(t\right) &= C\left(t, r_t\right)x\left(t\right) + C_d\left(t, r_t\right)x\left(t - d\left(t\right)\right) + D\left(r_t\right)\omega\left(t\right) \\ z\left(t\right) &= L\left(t, r_t\right)x\left(t\right) + L_d\left(t, r_t\right)x\left(t - d\left(t\right)\right) \\ x\left(t\right) &= \phi\left(t\right), \quad t \in \left[-d_2, 0\right] \end{aligned} \tag{2.1}$$

where $x\left(t\right) \in \mathbb{R}^n$ is the system state vector, $y\left(t\right) \in \mathbb{R}^m$ is the measured output, $z\left(t\right) \in \mathbb{R}^p$ is the signal to be estimated, $\omega\left(t\right) \in \mathbb{R}^q$ is the noise disturbance. The matrix $E\left(r_t\right) \in \mathbb{R}^{n\times n}$ may be singular, and it is assumed that $rank\left(E\left(r_t\right)\right) = r \le n$. $\phi\left(t\right)$ is a vector-valued initial continuous function defined on the interval $\left[-d_2, 0\right]$. $\left\{r_t, t \ge 0\right\}$ is a continuous-time discrete state Markov process with right continuous trajectory values in a finite set $\mathcal{S} = \left\{1, 2, \cdots, s\right\}$. For notation concision, for each $r_t = i \in \mathcal{S}$, a matrix $E\left(r_t\right)$ will be represented by E_i, and the parameters in system (1) are described as following:

$$\begin{array}{lll} A\left(t,r_t\right) = A_i + \Delta A_i\left(t\right) & A_d\left(t,r_t\right) = A_{di} + \Delta A_{di}\left(t\right) & B\left(r_t\right) = B_i \\ C\left(t,r_t\right) = C_i + \Delta C_i\left(t\right) & C_d\left(t,r_t\right) = C_{di} + \Delta C_{di}\left(t\right) & D\left(r_t\right) = D_i \\ L\left(t,r_t\right) = L_i + \Delta L_i\left(t\right) & L_d\left(t,r_t\right) = L_{di} + \Delta L_{di}\left(t\right) & \end{array} \tag{2.2}$$

Where $E_i, A_i, A_{di}, B_i, C_i, C_{di}, D_i, L_i$ and L_{di} are known real matrices with appropriate dimensions for each $i \in \mathcal{S}$, $\Delta A_i\left(t\right), \Delta A_{di}\left(t\right), \Delta C_i\left(t\right), \Delta C_{di}\left(t\right), \Delta L_i\left(t\right)$ and $\Delta L_{di}\left(t\right)$ are unknown matrices representing the parameter uncertainties of the systems, which are assumed to be of the form:

$$\left[\begin{array}{cc} \Delta A_i\left(t\right) & \Delta A_{di}\left(t\right) \\ \Delta C_i\left(t\right) & \Delta C_{di}\left(t\right) \\ \Delta L_i\left(t\right) & \Delta L_{di}\left(t\right) \end{array}\right] = \left[\begin{array}{c} \tilde{M}_{1i} \\ \tilde{M}_{2i} \\ \tilde{M}_{3i} \end{array}\right] F_i\left(t\right) \left[\begin{array}{cc} \tilde{N}_{1i} & \tilde{N}_{2i} \end{array}\right] \tag{2.3}$$

Where $\tilde{M}_{1i}, \tilde{M}_{2i}, \tilde{M}_{3i}, \tilde{N}_{1i}$ and $\tilde{N}_{2i}$ are known constant matrices with compatible dimensions and $F_i\left(t\right)$ are the uncertain matrices satisfying

$$F_i^T\left(t\right) F_i\left(t\right) \le I, \quad \forall i \in \mathcal{S} \tag{2.4}$$

Remark 2.1 The matrices $\Delta A_i\left(t\right), \Delta A_{di}\left(t\right), \Delta C_i\left(t\right), \Delta C_{di}\left(t\right), \Delta L_i\left(t\right)$ and $\Delta L_{di}\left(t\right)$ are said to be admissible if and only if both (3) and (4) hold. The parameter uncertainty structure as in (3) is an extension of the so-called matching condition, which has been widely used in the problems of control and robust filtering of uncertain dynamic systems.

In this paper, time-varying delay $d\left(t\right)$ is a differentiable function, satisfying for all $t \ge 0$

$$d_1 \le d\left(t\right) \le d_2 \qquad \dot{d}\left(t\right) \le \mu \tag{2.5}$$

Where d_1, d_2 and μ are known positive constants.

The transition probability matrix $\Pi = \pi_{ij}\left(i, j \in \mathcal{S}\right)$ is given by

$$\Pr\left\{r_{t+\Delta} = j \left| r_t = i\right.\right\} = \left\{\begin{array}{ll} \pi_{ij}\Delta + o\left(\Delta\right) & if \quad i \ne j \\ 1 + \pi_{ii}\Delta + o\left(\Delta\right) & if \quad i = j \end{array}\right. \tag{2.6}$$

Where $\Delta > 0$, $\lim\limits_{\Delta \to 0}\left(o\left(\Delta\right)/\Delta\right) = 0, \pi_{ij} \ge 0$ for $j \ne i$ is the transition rate from mode i at time t to mode j at time $t + \Delta$ and $\pi_{ii} = -\sum\limits_{\substack{j=1 \\ j \ne i}}^{s} \pi_{ij}$. In this paper, the aim is to design a filter to estimate $z\left(t\right)$, which is described as

$$\begin{array}{r} E_i\dot{x}_f\left(t\right) = A_{fi}x_f\left(t\right) + B_{fi}y\left(t\right) \\ z_f\left(t\right) = C_{fi}x_f\left(t\right) + D_{fi}y\left(t\right) \end{array} \tag{2.7}$$

Where $x_f\left(t\right) \in \mathbb{R}^n$ is the filter state vector, A_{fi}, B_{fi}, C_{fi} and D_{fi} are filter parameters with appropriate dimension to be determined.

Connecting filter (7) to system (1), we can have the filtering error system:

$$\begin{array}{r} \hat{E}\dot{x}\left(t\right) = \hat{A}_i\left(t\right)\hat{x}\left(t\right) + \hat{A}_{di}\left(t\right)\hat{x}\left(t - d\left(t\right)\right) + \hat{B}_i\omega\left(t\right) \\ \hat{z}\left(t\right) = \hat{L}_i\left(t\right)\hat{x}\left(t\right) + \hat{L}_{di}\left(t\right)\hat{x}\left(t - d\left(t\right)\right) - D_{fi}D_i\omega\left(t\right) \end{array} \tag{2.8}$$

Where

$$\hat{x}(t) = \begin{bmatrix} x(t) \\ x_f(t) \end{bmatrix} \; \hat{A}_i(t) = \hat{A}_i + \Delta\hat{A}_i(t) = \begin{bmatrix} A_i & 0 \\ B_{fi}C_i & A_{fi} \end{bmatrix} + \begin{bmatrix} \Delta A_i(t) & 0 \\ B_{fi}\Delta C_i(t) & 0 \end{bmatrix} \; \hat{B}_i = \begin{bmatrix} B_i \\ B_{fi}D_i \end{bmatrix}$$

$$\hat{A}_{di}(t) = \hat{A}_{di} + \Delta\hat{A}_{di}(t) = \begin{bmatrix} A_{di} & 0 \\ B_{fi}C_{di} & 0 \end{bmatrix} + \begin{bmatrix} \Delta A_{di}(t) & 0 \\ B_{fi}\Delta C_{di}(t) & 0 \end{bmatrix} \quad \hat{E} = \begin{bmatrix} E_i & 0 \\ 0 & E_i \end{bmatrix}$$

$$\hat{L}_i(t) = \hat{L}_i + \Delta\hat{L}_i(t) = \begin{bmatrix} L_i - D_{fi}C_i & -C_{fi} \end{bmatrix} + \begin{bmatrix} \Delta L_i(t) - D_{fi}C_i(t) & 0 \end{bmatrix} \quad \hat{z}(t) = z(t) - z_f(t)$$

$$\hat{L}_{di}(t) = \hat{L}_{di} + \Delta\hat{L}_{di}(t) = \begin{bmatrix} C_{di} - D_{fi}C_{di} & 0 \end{bmatrix} + \begin{bmatrix} \Delta L_{di}(t) - D_{fi}\Delta C_{di}(t) & 0 \end{bmatrix}$$

Now, some definitions are needed throughout this paper.

Definition 2.1. *[34]*

(1). Filtering error system (8) with $\omega(t) \equiv 0$ is said to be regular and impulse-free for time-varying delay if pairs $\left(\hat{E}, \hat{A}_i(t)\right)$ and $\left(\hat{E}, \hat{A}_i(t) + \hat{A}_{di}(t)\right)$ are regular and impulse-free for $\forall i \in \mathcal{S}$.

(2). Filtering error system (8) with $\omega(t) \equiv 0$ is said to be exponentially mean-square stable, if there exist scalars $a > 0$ and $b > 0$ such that

$$E\left\{ \|\hat{x}(t)\|^2 \left| \hat{\phi}(t), r_0 \right. \right\} \le a e^{-bt} \left\| \hat{\phi}(t) \right\|_{d_2}^2 \tag{2.9}$$

for any initial conditions $\hat{\phi}(t) \in \mathbb{R}^{2n}$ and $r_0 \in \mathcal{S}$.

(3). Filtering error system (8) with $\omega(t) \equiv 0$ is said to be exponentially mean-square admissible, if it is regular, impulse-free and exponentially mean-square stable for all admissible uncertainty in (2).

Definition 2.2. *[16] Given a scalar $\gamma > 0$, the filtering error system (8) is said to be robustly exponentially mean-square admissible with an H_∞ disturbance attenuation performance index γ if it is robustly exponentially mean-square admissible and any nonzero $\omega(t)$, satisfies*

$$E\left\{ \int_0^\infty \hat{z}^T(t)\,\hat{z}(t)\,dt \right\} < \gamma^2 \int_0^\infty \omega^T(t)\,\omega(t)\,dt \tag{2.10}$$

Before proceeding further, we will introduce the following lemmas which will be played an important role in the derivation of our main results.

Lemma 2.1. *[35] For any constant matrix $M > 0$, any scalars a and b with $a < b$, and a vector function $x(t) : [a, b] \to \mathbb{R}^n$ such that the integrals concerned are well defined, then the following inequality holds:*

$$\left[\int_a^b x(s)\,ds\right]^T M \left[\int_a^b x(s)\,ds\right] \le (b-a) \int_a^b x^T(s)\,M x(s)\,ds \tag{2.11}$$

Lemma 2.2. *[37] For a given symmetric positive definite matrix P, and any differentiable function ϖ in $[a, b] \to \mathbb{R}^n$, then the following inequality holds:*

$$\int_a^b \dot{\varpi}^T(s)\,P\dot{\varpi}(s)\,ds \ge \tfrac{1}{b-a}\left[(\varpi(b) - \varpi(a))^T P (\varpi(b) - \varpi(a))\right]$$
$$+ \tfrac{3}{b-a}\left[\left(\varpi(b) + \varpi(a) - \tfrac{2}{b-a}\int_a^b \varpi(s)\,ds\right)\right]^T P \left[\left(\varpi(b) + \varpi(a) - \tfrac{2}{b-a}\int_a^b \varpi(s)\,ds\right)\right] \tag{2.12}$$

Lemma 2.3. *[33] Let M, N and F be real matrices of appropriate dimensions with $F_i^T(t)\,F_i(t) \le I$, where F may be time-varying. Then, for any scalars $\varepsilon \ne 0$, we have*

$$MFN + (MFN)^T \le \varepsilon M M^T + \varepsilon^{-1} N^T N \tag{2.13}$$

Lemma 2.4. *[32] For any vectors $X, Y \in \mathbb{R}^n$, and positive definite matrix $Q \in \mathbb{R}^{n\times n}$, the following inequality holds:*

$$2X^TY \le X^TQX + Y^TQ^{-1}Y \tag{2.14}$$

Lemma 2.5. *[36] Let $Q(x) = Q^T(x), R(x) = R^T(x)$, and $S(x)$ depend affinely on x. Then the following linear matrix inequality*

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} > 0$$

holds if and only if one of the following conditions satisfies:
(1) $R(x) > 0, Q(x) - S(x)R^{-1}(x)S^T(x) > 0$;
(2) $Q(x) > 0, R(x) - S^T(x)Q^{-1}(x)S(x) > 0$.

Lemma 2.6. *[15] Suppose that a positive continuous function $f(t)$ satisfies*

$$f(t) \le \varsigma_1 \sup_{-d_2\le s\le t} f(s) + \varsigma_2 e^{-\varepsilon t} \tag{2.15}$$

where $\varepsilon > 0$, $\varsigma_1 < 1$, $\varsigma_2 > 0$. Then, $f(t)$ satisfies

$$f(t) \le \left(\sup_{-d_2\le s\le t} f(s) + \frac{\varsigma_2}{1-\varsigma_1 e^{\xi_0 d_2}} \right) e^{-\xi_0 t} \qquad t \ge 0 \tag{2.16}$$

3 Design of the H_∞ Filter

In this section, we will analyze the exponentially mean-square admissible with an H_∞ performance level $\gamma > 0$ of the filtering error system (8) at first, and give the following result.

Theorem 3.1. *Given a scalar $\gamma > 0$, the filtering error system (8) is exponentially mean-square admissible with an H_∞ performance level γ if there exist symmetric positive definite matrices $P_i, Q, Q_1, Q_2, Q_3, Q_{3i}, S, S_i$, Z_1, Z_2, Z_3, and any real matrices M_k, N_k $(k = 1, 2, \cdots, 7)$, T_i such that for every $i \in \mathcal{S}$ the following LMIs hold:*

$$\sum_{j=1}^{s} \pi_{ij} Q_{3j} - Q < 0 \tag{3.1}$$

$$\sum_{j=1}^{s} \pi_{ij} S_j - S < 0 \tag{3.2}$$

$$\Phi_i = \begin{bmatrix}
\Phi_{i11} & \Phi_{i12} & \Phi_{i13} & \Phi_{i14} & \Phi_{i15} & \hat{E}^T N_6^T & \hat{E}^T N_7^T & \Phi_{i18} & \hat{L}_i^T(t) & \sqrt{d_2}N_1 & \sqrt{d}M_1 \\
* & \Phi_{i22} & \Phi_{i23} & \Phi_{i24} & \Phi_{i25} & \Phi_{i26} & \Phi_{i27} & \Phi_{i28} & \hat{L}_{di}^T(t) & \sqrt{d_2}N_2 & \sqrt{d}M_2 \\
* & * & \Phi_{i33} & -M_3\hat{E} & \Phi_{i35} & \Phi_{i36} & 0 & 0 & 0 & \sqrt{d_2}N_3 & \sqrt{d}M_3 \\
* & * & * & \Phi_{i44} & -\hat{E}^T M_5^T & -\hat{E}^T M_6^T & \Phi_{i47} & 0 & 0 & \sqrt{d_2}N_4 & \sqrt{d}M_4 \\
* & * & * & * & \Phi_{i55} & 0 & 0 & 0 & 0 & \sqrt{d_2}N_5 & \sqrt{d}M_5 \\
* & * & * & * & * & \Phi_{i66} & 0 & 0 & 0 & \sqrt{d_2}N_6 & \sqrt{d}M_6 \\
* & * & * & * & * & * & \Phi_{i77} & 0 & 0 & \sqrt{d_2}N_7 & \sqrt{d}M_7 \\
* & * & * & * & * & * & * & \Phi_{i88} & \Phi_{i89} & 0 & 0 \\
* & * & * & * & * & * & * & * & -I & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -Z_3 & 0 \\
* & * & * & * & * & * & * & * & * & * & -Z_3
\end{bmatrix} < 0 \tag{3.3}$$

Where

$$\Phi_{i11} = \left(\hat{E}^T P_i + T_i\hat{R}_i\right)\hat{A}_i(t) + \hat{A}_i^T(t)\left(\hat{E}^T P_i + T_i\hat{R}_i\right)^T + \hat{E}^T\bar{P}\hat{E} + d_2Q + Q_1 + Q_2 + d_1Q_3 + Q_{3i} + dS_i + \frac{1}{2}\left(d_2^2 - d_1^2\right)S - \frac{4}{d_1}\hat{E}^T Z_1\hat{E} + N_1\hat{E} + \hat{E}^T N_1^T + \hat{A}_i^T(t)Z\hat{A}_i(t)$$

$$\Phi_{i12} = \left(\hat{E}^T P_i + T_i\hat{R}_i\right)\hat{A}_{di}(t) - N_1\hat{E} + \hat{E}^T N_2^T + M_1\hat{E} + \hat{A}_i^T(t)Z\hat{A}_{di}(t)$$

$\Phi_{i14} = \hat{E}^T N_4^T - M_1\hat{E} \quad \Phi_{i15} = \frac{6}{d_1^2}\hat{E}^T Z_1\hat{E} + \hat{E}^T N_5^T \quad \Phi_{i18} = \left(\hat{E}^T P_i + T_i\hat{R}_i\right)\hat{B}_i + \hat{A}_i^T(t) Z\hat{B}_i$
$\Phi_{i22} = -(1-\mu) Q_{3i} - \frac{8}{d}\hat{E}^T Z_2\hat{E} - N_2\hat{E} - \hat{E}^T N_2^T + M_2\hat{E} + \hat{E}^T M_2^T + \hat{A}_{di}^T(t) Z\hat{A}_{di}(t)$
$\Phi_{i23} = -\frac{2}{d}\hat{E}^T Z_2\hat{E} - \hat{E}^T N_3^T + \hat{E}^T M_3^T \quad \Phi_{i24} = -\frac{2}{d}\hat{E}^T Z_2\hat{E} - \hat{E}^T N_4^T - M_2\hat{E} + \hat{E}^T M_4^T$
$\Phi_{i25} = -\hat{E}^T N_5^T + \hat{E}^T M_5^T \quad \Phi_{i26} = \frac{6}{d^2}\hat{E}^T Z_2\hat{E} - \hat{E}^T N_6^T + \hat{E}^T M_6^T$
$\Phi_{i27} = \frac{6}{d^2}\hat{E}^T Z_2\hat{E} - \hat{E}^T N_7^T + \hat{E}^T M_7^T \quad \Phi_{i33} = -Q_1 - \frac{4}{d_1}\hat{E}^T Z_1\hat{E} - \frac{4}{d}\hat{E}^T Z_2\hat{E}$
$\Phi_{i35} = \frac{6}{d_1^2}\hat{E}^T Z_1\hat{E} \quad \Phi_{i36} = \frac{6}{d^2}\hat{E}^T Z_2\hat{E} \quad \Phi_{i44} = -Q_2 - \frac{4}{d}\hat{E}^T Z_2\hat{E} - M_4\hat{E} - \hat{E}^T M_4^T$
$\Phi_{i47} = \frac{6}{d^2}\hat{E}^T Z_2\hat{E} - \hat{E}^T M_7^T \quad \Phi_{i55} = -\frac{12}{d_1^3}\hat{E}^T Z_1\hat{E} - \frac{1}{d_1}Q_3 \quad \Phi_{i66} = -\frac{1}{d}S_i - \frac{12}{d^3}\hat{E}^T Z_2\hat{E}$
$\Phi_{i77} = -\frac{12}{d^3}\hat{E}^T Z_2\hat{E} - \frac{1}{d}S_i \quad \Phi_{i89} = -D_i^T D_{fi}^T \quad \Phi_{i13} = -\frac{2}{d_1}\hat{E}^T Z_1\hat{E} + \hat{E}^T N_3^T$
$\Phi_{i28} = \hat{A}_{di}^T(t) Z\hat{B}_i \quad \Phi_{i88} = -\gamma^2 I + \hat{B}_i^T Z\hat{B}_i \quad d = d_2 - d_1 \quad \bar{P} = \sum_{j=1}^{s}\pi_{ij}P_j$
$Z = d_1 Z_1 + dZ_2 + d_2 Z_3 \quad \hat{R}\hat{E} = 0\left(\hat{R} \in \mathbb{R}^{2(n-r)\times 2n}\right)$

Proof. Firstly, we will show that the filtering error system (8) is regular and impulse-free. From (19), it is seen that

$$\left(\hat{E}^T P_i + T_i\hat{R}_i\right)\hat{A}_i(t) + \hat{A}_i^T(t)\left(\hat{E}^T P_i + T_i\hat{R}_i\right)^T + \hat{E}^T\bar{P}\hat{E} - \frac{4}{d_1}\hat{E}^T Z_1\hat{E} + N_1\hat{E} + \hat{E}^T N_1^T < 0 \quad (3.4)$$

It is known that there exist two nonsingular matrices G and H, such that

$$G\hat{E}H = \begin{bmatrix} I_{2r} & 0 \\ 0 & 0 \end{bmatrix} \quad G^{-T}P_iG^{-1} = \begin{bmatrix} P_{1i} & P_{2i} \\ * & P_{3i} \end{bmatrix} \quad G\hat{A}_i(t)H = \begin{bmatrix} \hat{A}_{i11} & \hat{A}_{i12} \\ \hat{A}_{i21} & \hat{A}_{i22} \end{bmatrix}$$
$$H^T T_i = \begin{bmatrix} T_{1i} \\ T_{2i} \end{bmatrix} \quad \hat{R} = \begin{bmatrix} 0 & U \end{bmatrix} G \quad G\hat{A}_{di}(t)H = \begin{bmatrix} \hat{A}_{di11} & \hat{A}_{di12} \\ \hat{A}_{di21} & \hat{A}_{di22} \end{bmatrix} \quad H^T Q_{3i} H = \begin{bmatrix} Q_{3i11} & Q_{3i12} \\ * & Q_{3i22} \end{bmatrix}$$

Where U is any real nonsingular matrix. Now, we will Pre-multiplying and Post-multiplying the (20) by H^T and H, we can have

$$T_{2i}U\hat{A}_{i22} + \hat{A}_{i22}^T U^T T_{2i}^T < 0 \quad (3.5)$$

which implies A_{i22} is nonsingular. Then, for $i \in \mathcal{S}$, pair $\begin{pmatrix} \hat{E} & \hat{A}_i(t) \end{pmatrix}$ is regular and impulse-free. Moreover, from (19), we can obtain that

$$\begin{bmatrix} I \\ I \end{bmatrix}^T \begin{bmatrix} \Phi_{i11} & \Phi_{i12} \\ * & \Phi_{i22} \end{bmatrix} \begin{bmatrix} I \\ I \end{bmatrix} < 0 \quad (3.6)$$

which implies

$$\left(\hat{A}_i(t) + \hat{A}_{di}(t)\right)\left(\hat{E}^T P_i + T_i\hat{R}\right) + \left(\left(\hat{A}_i(t) + \hat{A}_{di}(t)\right)\left(\hat{E}^T P_i + T_i\hat{R}\right)\right)^T + \hat{E}^T\bar{P}\hat{E}$$
$$M_1\hat{E} + \hat{E}^T M_1^T + M_2\hat{E} + \hat{E}^T M_2^T - \frac{4}{d_1}\hat{E}^T Z_1\hat{E} - \frac{8}{d}\hat{E}^T Z_2\hat{E} < 0 \quad (3.7)$$

Similar to (20), it is concluded from (23) that pair $\begin{pmatrix} \hat{E} & \hat{A}_i(t) + \hat{A}_{di}(t) \end{pmatrix}$ is regular and impulse-free for $i \in \mathcal{S}$ is regular and impulse-free for $i \in \mathcal{S}$. Then, filtering error system (8) is regular and impulse-free for any time-varying $d(t)$ satisfying (5).

Now, we will show filtering error system (8) with $\omega(t) \equiv 0$ is exponentially mean-square stable. Now, choose a stochastic Lyapunov function candidate for the filtering error system (8):

$$V_1(r_t, i, t) = \hat{x}^T(t)\hat{E}^T P(r_t)\hat{E}\hat{x}(t) \quad (3.8)$$

$$V_2(r_t,i,t)=\int_{t-d_1}^{t}\hat{x}^T(s)\,Q_1\hat{x}(s)\,ds+\int_{t-d_2}^{t}\hat{x}^T(s)\,Q_2\hat{x}(s)\,ds\int_{t-d(t)}^{t}\hat{x}^T(s)\,Q_3(r_t)\,\hat{x}(s)\,ds \quad (3.9)$$

$$V_3(r_t,i,t)=\int_{-d_1}^{0}\int_{t+\theta}^{t}\hat{x}^T(s)\,Q_3\hat{x}(s)\,dsd\theta+\int_{-d_2}^{0}\int_{t+\theta}^{t}\hat{x}^T(s)\,Q\hat{x}(s)\,dsd\theta \quad (3.10)$$

$$V_4(r_t,i,t)=\int_{-d_2}^{-d_1}\int_{t+\theta}^{t}\hat{x}^T(s)\,S(r_t)\,\hat{x}(s)\,dsd\theta+\int_{-d_2}^{-d_1}\int_{\theta}^{0}\int_{t+v}^{t}\hat{x}^T(s)\,S\hat{x}(s)dsdvd\theta \quad (3.11)$$

$$V_5(r_t,i,t)=\int_{-d_1}^{0}\int_{t+\theta}^{t}\dot{\hat{x}}^T(s)\,\hat{E}^TZ_1\hat{E}\dot{\hat{x}}(s)\,dsd\theta+\int_{-d_2}^{-d_1}\int_{t+\theta}^{t}\dot{\hat{x}}^T(s)\,\hat{E}^TZ_2\hat{E}\dot{\hat{x}}(s)\,dsd\theta$$
$$+\int_{-d_2}^{0}\int_{t+\theta}^{t}\dot{\hat{x}}^T(s)\,\hat{E}^TZ_3\hat{E}\dot{\hat{x}}(s)\,dsd\theta \quad (3.12)$$

Then, we define the following weak infinitesimal operator L:

$$LV(r_t,i,t)=\lim_{\Delta\to 0}\frac{1}{\Delta}\left[E\left\{V\left((x_{t+\Delta},\delta(t+\Delta),t+\Delta\right)|r_t=i\right)-V(r_t,i,t)\}\right] \quad (3.13)$$

So, it can be deduced that for each $i\in\mathcal{S}$

$$\begin{aligned}
LV_1(r_t,i,t)&=2\hat{x}^T(t)\,\hat{E}^TP_i\hat{E}\dot{\hat{x}}(t)+\hat{x}^T(t)\,\hat{E}^T\bar{P}\hat{E}\hat{x}(t)+2\hat{x}^T(t)\,T_i\hat{R}_i\hat{E}\dot{\hat{x}}(t)\\
&=2\hat{x}^T(t)\left(\hat{E}^TP+T_i\hat{R}_i\right)\left(\hat{A}_i(t)\,\hat{x}(t)+\hat{A}_{di}(t)\,\hat{x}(t-d(t))\right)+\hat{x}^T(t)\,\hat{E}^T\bar{P}\hat{E}\hat{x}(t)\\
&=\hat{x}^T(t)\left[\left(\hat{E}^TP+T_i\hat{R}_i\right)\hat{A}_i(t)+\hat{A}_i^T(t)\left(\hat{E}^TP+T_i\hat{R}_i\right)^T\right]\hat{x}(t)+\hat{x}^T(t)\,\hat{E}^T\bar{P}\hat{E}\hat{x}(t)\\
&+2\hat{x}^T(t)\left(\hat{E}^TP+T_i\hat{R}_i\right)\hat{A}_{di}(t)\,\hat{x}(t-d(t))\\
LV_2(r_t,i,t)&\le\hat{x}^T(t)\,(Q_1+Q_2+Q_{3i})\,\hat{x}(t)-\hat{x}^T(t-d_1)\,Q_1\hat{x}(t-d_1)-\hat{x}^T(t-d_2)\,Q_1\hat{x}(t-d_2)\\
&-(1-\mu)\,\hat{x}^T(t-d(t))\,Q_{3i}\hat{x}(t-d(t))+\textstyle\int_{t-d(t)}^{t}\hat{x}^T(s)\sum_{j=1}^{s}\pi_{ij}Q_{3j}\hat{x}(s)\,ds\\
LV_3(r_t,i,t)&\le d_2\hat{x}^T(t)\,Q\hat{x}(t)+d_1Q_3-\textstyle\int_{t-d(t)}^{t}\hat{x}^T(s)\,Q\hat{x}(s)\,ds-\frac{1}{d_1}\int_{t-d_1}^{t}\hat{x}^T(s)\,dsQ_3\int_{t-d_1}^{t}\hat{x}(s)\,ds\\
LV_4(r_t,i,t)&=d\hat{x}^T(t)\,S_i\hat{x}(t)+\tfrac{1}{2}\left(d_2^2-d_1^2\right)\hat{x}^T(t)\,S\hat{x}(t)-\textstyle\int_{t-d_2}^{t-d_1}\hat{x}^T(s)\,S_i\hat{x}(s)\,ds\\
&+\textstyle\int_{-d_2}^{-d_1}\int_{t+\theta}^{t}\hat{x}^T(s)\sum_{j=1}^{s}\pi_{ij}S_j\hat{x}(s)\,ds-\int_{-d_2}^{-d_1}\int_{t+\theta}^{t}\hat{x}^T(s)\,S\hat{x}(s)\,ds
\end{aligned}$$

$$-\textstyle\int_{t-d_2}^{t-d_1}\hat{x}^T(s)\,S_i\hat{x}(s)\,ds=-\int_{t-d_2}^{t-d(t)}\hat{x}^T(s)\,S_i\hat{x}(s)\,ds-\int_{t-d(t)}^{t-d_1}\hat{x}^T(s)\,S_i\hat{x}(s)\,ds$$
$$\le-\frac{1}{d}\int_{t-d_2}^{t-d(t)}\hat{x}^T(s)\,dsS_i\int_{t-d_2}^{t-d(t)}\hat{x}(s)\,ds-\frac{1}{d}\int_{t-d(t)}^{t-d_1}\hat{x}^T(s)\,dsS_i\int_{t-d(t)}^{t-d_1}\hat{x}(s)\,ds$$

$$\begin{aligned}
LV_5(r_t,i,t)&=\dot{\hat{x}}^T(t)\,\hat{E}^TZ\hat{E}\dot{\hat{x}}(t)-\textstyle\int_{t-d_1}^{t}\dot{\hat{x}}^T(s)\,\hat{E}^TZ_1\hat{E}\dot{\hat{x}}(s)\,ds-\int_{t-d_2}^{t-d(t)}\dot{\hat{x}}^T(s)\,\hat{E}^TZ_2\hat{E}\dot{\hat{x}}(s)\,ds\\
&-\textstyle\int_{t-d(t)}^{t-d_1}\dot{\hat{x}}^T(s)\,\hat{E}^TZ_2\hat{E}\dot{\hat{x}}(s)\,ds-\int_{t-d_2}^{t}\dot{\hat{x}}^T(s)\,\hat{E}^TZ_3\hat{E}\dot{\hat{x}}(s)\,ds
\end{aligned}$$

Now, from the Lemma (2.2), we can have

$$-\textstyle\int_{t-d_1}^{t}\dot{\hat{x}}^T(s)\,\hat{E}^TZ_1\hat{E}\dot{\hat{x}}(s)\,ds\le-\frac{1}{d_1}\left[(\hat{x}(t)-\hat{x}(t-d_1))^T\,\hat{E}^TZ_1\hat{E}\,(\hat{x}(t)-\hat{x}(t-d_1))\right]$$

$$-\frac{3}{d_1}\left[\hat{x}(t)+\hat{x}(t-d_1)-\frac{2}{d_1}\int_{t-d_1}^{t}\hat{x}(s)\,ds\right]^T\hat{E}^TZ_1\hat{E}\left[\hat{x}(t)+\hat{x}(t-d_1)-\frac{2}{d_1}\int_{t-d_1}^{t}\hat{x}(s)\,ds\right] \quad (3.14)$$

$$-\textstyle\int_{t-d_2}^{t-d(t)}\dot{\hat{x}}^T(s)\,\hat{E}^TZ_2\hat{E}\dot{\hat{x}}(s)\,ds\le-\frac{1}{d}\left[(\hat{x}(t-d(t))-\hat{x}(t-d_2))^T\,\hat{E}^TZ_2\hat{E}\,(\hat{x}(t-d(t))-\hat{x}(t-d_2))\right]$$

$$-\frac{3}{d}\left[\hat{x}\left(t-d\left(t\right)\right)+\hat{x}\left(t-d_2\right)-\frac{2}{d}\int_{t-d_2}^{t-d(t)}\hat{x}\left(s\right)ds\right]^T\hat{E}^TZ_2\hat{E}\left[\hat{x}\left(t-d\left(t\right)\right)+\hat{x}\left(t-d_2\right)-\frac{2}{d}\int_{t-d_2}^{t-d(t)}\hat{x}\left(s\right)ds\right] \tag{3.15}$$

$$-\int_{t-d(t)}^{t-d_1}\dot{\hat{x}}^T\left(s\right)\hat{E}^TZ_2\hat{E}\dot{\hat{x}}\left(s\right)ds\le-\tfrac{1}{d}\left[\left(\hat{x}\left(t-d_1\right)-\hat{x}\left(t-d\left(t\right)\right)\right)^T\hat{E}^TZ_2\hat{E}\left(\hat{x}\left(t-d_1\right)-\hat{x}\left(t-d\left(t\right)\right)\right)\right]$$

$$-\frac{3}{d}\left[\hat{x}\left(t-d\left(t\right)\right)+\hat{x}\left(t-d_1\right)-\frac{2}{d}\int_{t-d(t)}^{t-d_1}\hat{x}\left(s\right)ds\right]^T\hat{E}^TZ_2\hat{E}\left[\hat{x}\left(t-d\left(t\right)\right)+\hat{x}\left(t-d_1\right)-\frac{2}{d}\int_{t-d(t)}^{t-d_1}\hat{x}\left(s\right)ds\right] \tag{3.16}$$

Next, it is seen that two equations as follows are held for the any matrices N and M

$$2\zeta^T\left(t\right)N\left[\hat{E}\hat{x}\left(t\right)-\hat{E}\hat{x}\left(t-d\left(t\right)\right)-\int_{t-d(t)}^{t}\hat{E}\dot{\hat{x}}\left(s\right)ds\right]=0 \tag{3.17}$$

$$2\zeta^T\left(t\right)M\left[\hat{E}\hat{x}\left(t-d\left(t\right)\right)-\hat{E}\hat{x}\left(t-d_2\right)-\int_{t-d_2}^{t-d(t)}\hat{E}\dot{\hat{x}}\left(s\right)ds\right]=0 \tag{3.18}$$

Where $N=\left[\begin{array}{ccccccc} N_1^T & N_2^T & N_3^T & N_4^T & N_5^T & N_6^T & N_7^T\end{array}\right]^T$

$M=\left[\begin{array}{ccccccc} M_1^T & M_2^T & M_3^T & M_4^T & M_5^T & M_6^T & M_7^T\end{array}\right]^T$

$\zeta\left(t\right)=\left[\hat{x}^T\left(t\right),\hat{x}^T\left(t-d\left(t\right)\right),\hat{x}^T\left(t-d_1\right),\hat{x}^T\left(t-d_2\right),\int_{t-d_1}^{t}\hat{x}^T\left(s\right)ds,\int_{t-d(t)}^{t-d_1}\hat{x}^T\left(s\right)ds,\int_{t-d_2}^{t-d(t)}\hat{x}^T\left(s\right)ds\right]^T$

Now, from the Lemma (2.4), we can obtain that

$$-2\zeta^T\left(t\right)N\int_{t-d(t)}^{t}\hat{E}\dot{\hat{x}}\left(s\right)ds\le d_2\zeta^T\left(t\right)NZ_3^{-1}N^T\zeta\left(t\right)+\int_{t-d(t)}^{t}\dot{\hat{x}}^T\left(s\right)\hat{E}^TZ_3\hat{E}\dot{\hat{x}}\left(s\right)ds \tag{3.19}$$

$$-2\zeta^T\left(t\right)M\int_{t-d_2}^{t-d(t)}\hat{E}\dot{\hat{x}}\left(s\right)ds\le d\zeta^T\left(t\right)MZ_3^{-1}M^T\zeta\left(t\right)+\int_{t-d_2}^{t-d(t)}\dot{\hat{x}}^T\left(s\right)\hat{E}^TZ_3\hat{E}\dot{\hat{x}}\left(s\right)ds \tag{3.20}$$

So, when $\omega\left(t\right)=0$, we obtain

$$LV\left(r_t,i,t\right)\le\zeta^T\left(t\right)\left(\Phi_i'+d_2NZ_3^{-1}N^T+dMZ_3^{-1}M^T\right)\zeta\left(t\right) \tag{3.21}$$

Then by the Schur complement Lemma (2.5), we can find that (19) guarantees

$$\Phi_i'+d_2NZ_3^{-1}N^T+dMZ_3^{-1}M^T<0$$

Thus, there exists a scalar $\delta>0$ such that for $i\in\mathcal{S}$

$$LV\left(r_t,i,t\right)<-\delta\left|\hat{x}\left(t\right)\right|^2 \tag{3.22}$$

Next, we will show that filtering error system (8) with $\omega\left(t\right)=0$ is exponentially mean-square stable. Because $\hat{A}_{i22}$ is nonsingular for $i\in\mathcal{S}$, then we introduce some new matrices as follows:

$$G\hat{E}H=\left[\begin{array}{cc} I_{2r} & 0\\ 0 & 0\end{array}\right]\qquad\bar{G}=\left[\begin{array}{cc} I & -\hat{A}_{i12}\hat{A}_{i22}\\ 0 & -\hat{A}_{i22}^{-1}\end{array}\right]\qquad\bar{G}\hat{E}H=\left[\begin{array}{cc} I_{2r} & 0\\ 0 & 0\end{array}\right]$$

$$\bar{G}\hat{A}_i\left(t\right)H=\left[\begin{array}{cc}\bar{A}_{i11} & 0\\ \bar{A}_{i21} & I\end{array}\right]\qquad\bar{G}\hat{A}_{di}\left(t\right)H=\left[\begin{array}{cc}\bar{A}_{di11} & \bar{A}_{di12}\\ \bar{A}_{di21} & \bar{A}_{di22}\end{array}\right] \tag{3.23}$$

where $\bar{A}_{i11} = \hat{A}_{i11} - \hat{A}_{i12}\hat{A}_{i22}^{-1}\hat{A}_{i21}, \bar{A}_{i21} = -\hat{A}_{i22}^{-1}\hat{A}_{i21}$

Set

$$\bar{x}(t) = \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} = H^{-1}\hat{x}(t)$$

then for all $i \in \mathcal{S}$, the filtering error system (8) with $\omega(t) = 0$ is rewritten as

$$\begin{array}{l} \dot{\hat{x}}_1(t) = \hat{A}_{i11}\hat{x}_1(t) + \hat{A}_{di11}\hat{x}_1(t - d(t)) + \hat{A}_{di12}\hat{x}_2(t - d(t)) \\ -\hat{x}_2(t) = \hat{A}_{i21}\hat{x}_1(t) + \hat{A}_{di21}\hat{x}_1(t - d(t)) + \hat{A}_{di22}\hat{x}_2(t - d(t)) \\ \bar{\phi}(t) = H^{-1}\phi(t), \quad t \in [-d_2, 0] \end{array} \tag{3.24}$$

Base on the definition of $V(r_t, i, t)$, we can obtain

$$V(r_t, i, t) \geq \hat{x}^T(t)\hat{E}^T P_i \hat{E}\hat{x}(t) = \bar{x}^T(t) H^T \hat{E}^T G^T G^{-T} P_i G^{-1} G\hat{E}H\bar{x}(t) \geq \kappa_1 \hat{x}_1^T(t)\hat{x}_1(t) \tag{3.25}$$

where $\kappa_1 = \min\limits_{i\in\mathcal{S}} \{\lambda_{\min}(P_{1i})\}$

Now, we consider the (38) and (41), which implies that

$$\int_0^T E\{L[V(r_t, i, t)]\}dt \leq \int_0^T E\left\{-\delta\hat{x}^T(t)\hat{x}(t)\right\}dt = -\delta\int_0^T E\left\{\begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix}^T H^T H \begin{bmatrix} \hat{x}_1(t) & \hat{x}_2(t) \end{bmatrix}\right\}dt \tag{3.26}$$

then, it is seen that

$$\kappa_1 E\left\{|\hat{x}_1(t)|^2 |\hat{x}_1(0), r_0\right\} \leq E\{V(\hat{x}(0), r_0)\} - \delta\lambda_{\min}\left(H^T H\right)\int_0^T E\left\{|\hat{x}_1(t)|^2\right\}dt \tag{3.27}$$

Now, by the Gronwall-Bellman Lemma to (43), we can obtain that there exist a positive scalar κ_2 such that

$$E\left\{|\hat{x}_1(t)|^2 |\hat{x}_1(0), r_0\right\} \leq \kappa_2 e^{-\frac{\delta\lambda_{\min}\left(H^T H\right)}{\kappa_1}t} |\phi(0)|^2 \tag{3.28}$$

which implies that $\hat{x}_1(t)$ is exponentially mean-square stable.

From the (19), we know that

$$\begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix}^T \begin{bmatrix} \Phi_{i11} & \Phi_{i12} \\ * & \Phi_{i22} \end{bmatrix} \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} < 0 \tag{3.29}$$

which implies that

$$\begin{bmatrix} T_{2i}U\hat{A}_{i22} + \hat{A}_{i22}^T U^T T_{2i}^T + Q_{3i22} & T_{2i}U\hat{A}_{i22} \\ * & -Q_{3i22} \end{bmatrix} < 0 \tag{3.30}$$

Similar to the [9], we can have $\kappa_3 \triangleq \max\limits_{i\in\mathcal{S}} \{|A_{id22}|\} < 1$ and from the (40), we can obtain

$$E\{|\hat{x}_2(t)|\} \leq \kappa_3 E\left\{\sup_{-d_2\leq s\leq t} |\hat{x}_2(s)|\right\} + \kappa_4 e^{-\kappa_5 t} \tag{3.31}$$

where

$$\kappa_4 = \sqrt{\kappa_2}\max_{i\in\mathcal{S}}\left\{\left|\hat{A}_{i21}\right|, \left|\hat{A}_{di21}\right|\right\}\phi(0)\left(1 + e^{\frac{d_2}{2}\times\frac{\delta\lambda_{\min}\left(H^T H\right)}{\kappa_1}}\right)$$

$$\kappa_5 = \min\left\{\frac{\frac{\delta\lambda_{\min}\left(H^T H\right)}{\kappa_1}}{2}, \kappa_6\left(\kappa_6 \in \left\{\kappa \left| 0 < \kappa_3 e^{\kappa d_2} < 1\right.\right\}\right)\right\}$$

By utilizing the Lemma 2.6, we can have

$$E\left\{\left|\hat{x}_2(t)\right|\right\} \leq \left(\sup_{-d_2\leq s\leq t}\left|\hat{x}_2(s)\right| + \frac{\kappa_4}{1-\kappa_3 e^{\kappa_5 d_2}}\right)e^{-\kappa_5 t} \tag{3.32}$$

which implies that $\hat{x}_2(t)$ is exponentially mean-square stable. So, at this time, we have concluded that the filtering error system (8) is exponentially mean-square admissible for $\forall i \in \mathcal{S}$.

Now, we will show that the filtering error system (8) with an H_∞ performance level γ. Thus, for the nonzero $\omega(t)$, we define the equality

$$G = E\left\{\int_0^\infty \left(z_f^T(t)\, z_f(t) - \gamma^2\omega^T(t)\,\omega(t)\right)dt\right\} \tag{3.33}$$

noting the zero initial condition and from the (19), it is easy to see that

$$\begin{aligned} G &\leq E\left\{\int_0^\infty \left(z_f^T(t)\, z_f(t) - \gamma^2\omega^T(t)\,\omega(t)\right)dt\right\} + E\left\{V\left(\hat{x}_\infty, r_\infty\right)\right\} - E\left\{V(0,0)\right\} \\ &= E\left\{\int_0^\infty \left[z_f^T(t)\, z_f(t) - \gamma^2\omega^T(t)\,\omega(t) + LV\left(r_t, i, t\right)\right]dt\right\} < 0 \end{aligned} \tag{3.34}$$

which implies that filtering error system (8) with an H_∞ performance level γ has been established. The proof is complete. □

Next, we will give a solution to the desired mode-dependent filtering for (7).

Theorem 3.2. *Given a scalar $\gamma > 0$, filtering error system (8) is robustly exponentially mean-square admissible with H_∞ performance over admissible uncertainty (3), if there exists positive matrices P_i, Q, Q_1, Q_2, Q_3,*
$Q_{3i}, S, S_i, Z_1, Z_2, Z_3$, positive scalar ε, and any real matrices $M_k, N_k\,(k = 1, 2, \cdots, 7)$, T_i , W_i, V_i, A_{Fi}, B_{Fi},
C_{Fi}, D_{Fi} such that for every $i \in \mathcal{S}$ the following LMIs hold:

$$\begin{bmatrix} \bar{Q}_{11}^3 - Q_{11} & \bar{Q}_{12}^3 - Q_{12} \\ * & \bar{Q}_{22}^3 - Q_{22} \end{bmatrix} < 0 \tag{3.35}$$

$$\begin{bmatrix} \bar{S}_{11} - S_{11} & \bar{S}_{12} - S_{12} \\ * & \bar{S}_{22} - S_{22} \end{bmatrix} < 0 \tag{3.36}$$

$$\begin{bmatrix} \Xi_i + \varepsilon\alpha_2\alpha_2^T & \alpha_1 \\ * & -\varepsilon \end{bmatrix} < 0 \tag{3.37}$$

where $\Xi_i = \Xi_{i12\times12}$

$$\begin{aligned}
\Xi_{i1\times1} &= W_i^T\hat{A}_i(t) + \hat{A}_i^T(t)\,W_i + \hat{E}^T\bar{P}\hat{E} + d_2Q + Q_1 + Q_2 + d_1Q_3 + Q_{3i} + dS_i + \tfrac{1}{2}\left(d_2^2 - d_1^2\right)S \\
&\quad - \tfrac{4}{d_1}\hat{E}^TZ_1\hat{E} + N_1\hat{E} + \hat{E}^TN_1^T
\end{aligned}$$

$\Xi_{i1\times2} = \hat{E}P_i + T_i\hat{R} - W_i^T + \hat{A}_i^TV_i \qquad \Xi_{i1\times3} = W_i^T\hat{A}_{di}^T - N_1\hat{E} + \hat{E}^TN_2^T + M_1\hat{E}$

$\Xi_{i1\times4} = -\frac{2}{d_1}\hat{E}^TZ_1\hat{E} + \hat{E}^TN_3^T \qquad \Xi_{i1\times5} = \hat{E}^TN_4^T - M_1\hat{E} \qquad \Xi_{i1\times6} = \frac{6}{d_1^2}\hat{E}^TZ_1\hat{E} + \hat{E}^TN_5^T$

$\Xi_{i1\times7} = \hat{E}^TN_6^T \qquad \Xi_{i1\times8} = \hat{E}^TN_7^T \qquad \Xi_{i1\times9} = W_i^T\hat{B}_i \qquad \Xi_{i1\times10} = \hat{L}_i^T \qquad \Xi_{i1\times11} = \sqrt{d_2}N_1$

$\Xi_{i1\times12} = \sqrt{d}M_1 \qquad \Xi_{i2\times2} = -V_i - V_i^T + Z \qquad \Xi_{i2\times3} = V_i^T\hat{A}_{di} \quad \Xi_{i2\times9} = V_i^T\hat{B}_i$

$\Xi_{i2\times4} = \Xi_{i2\times5} = \Xi_{i2\times6} = \Xi_{i2\times7} = \Xi_{i2\times8} = \Xi_{i2\times10} = \Xi_{i2\times11} = \Xi_{i2\times12} = 0$

$\Xi_{i3\times3} = -(1-\mu)\,Q_{3i} - \frac{8}{d}\hat{E}^TZ_2\hat{E} - N_2\hat{E} - \hat{E}^TN_2^T + M_2\hat{E} + \hat{E}^TM_2^T$

$\Xi_{i3\times4} = -\frac{2}{d}\hat{E}^TZ_2\hat{E} - \hat{E}^TN_3^T + \hat{E}^TM_3^T \qquad \Xi_{i3\times5} = -\frac{2}{d}\hat{E}^TZ_2\hat{E} - \hat{E}^TN_4^T - M_2\hat{E} + \hat{E}^TM_4^T$

$\Xi_{i3\times6} = -\hat{E}^TN_5^T + \hat{E}^TM_5^T \qquad \Xi_{i3\times7} = \frac{6}{d^2}\hat{E}^TZ_2\hat{E} - \hat{E}^TN_6^T + \hat{E}^TM_6^T$

$\Xi_{i3\times8} = \frac{6}{d^2}\hat{E}^TZ_2\hat{E} - \hat{E}^TN_7^T + \hat{E}^TM_7^T \qquad \Xi_{i3\times9} = 0 \qquad \Xi_{i3\times10} = L_{di}^T \qquad \Xi_{i3\times11} = \sqrt{d_2}N_2$

$\Xi_{i3\times12} = \sqrt{d}M_1 \qquad \Xi_{i4\times4} = -Q_1 - \frac{4}{d_1}\hat{E}^TZ_1\hat{E} - \frac{4}{d}\hat{E}^TZ_2\hat{E} \qquad \Xi_{i4\times5} = -M_3\hat{E} \qquad \Xi_{i4\times6} = \frac{6}{d_1^2}\hat{E}^TZ_1\hat{E}$

$\Xi_{i4\times7} = \frac{6}{d^2}\hat{E}^T Z_2 \hat{E} \quad \Xi_{i4\times8} = \Xi_{i4\times9} = \Xi_{i4\times10} = 0 \quad \Xi_{i4\times11} = \sqrt{d_2}N_3 \quad \Xi_{i4\times12} = \sqrt{d}M_3$

$\Xi_{i5\times5} = -Q_2 - \frac{4}{d}\hat{E}^T Z_2 \hat{E} - M_4\hat{E} - \hat{E}^T M_4^T \quad \Xi_{i5\times6} = -\hat{E}^T M_5^T \quad \Xi_{i5\times7} = -\hat{E}^T M_6^T$

$\Xi_{i5\times8} = \frac{6}{d^2}\hat{E}^T Z_2 \hat{E} - \hat{E}^T M_7^T \quad \Xi_{i5\times9} = \Xi_{i5\times10} = 0 \quad \Xi_{i5\times11} = \sqrt{d_2}N_4 \quad \Xi_{i5\times12} = \sqrt{d}M_4$

$\Xi_{i6\times6} = -\frac{12}{d_1^3}\hat{E}^T Z_1 \hat{E} - \frac{1}{d_1}Q_3 \quad \Xi_{i6\times7} = \Xi_{i6\times8} = \Xi_{i6\times9} = \Xi_{i6\times10} = 0 \quad \Xi_{i6\times11} = \sqrt{d_2}N_5$

$\Xi_{i6\times12} = \sqrt{d}M_5 \quad \Xi_{i7\times7} = -\frac{1}{d}S_i - \frac{12}{d^3}\hat{E}^T Z_2 \hat{E} \quad \Xi_{i7\times8} = \Xi_{i7\times9} = \Xi_{i7\times10} = 0 \quad \Xi_{i7\times11} = \sqrt{d_2}N_6$

$\Xi_{i7\times12} = \sqrt{d}M_6 \quad \Xi_{i8\times8} = -\frac{1}{d}S_i - \frac{12}{d^3}\hat{E}^T Z_2 \hat{E} \quad \Xi_{i8\times9} = \Xi_{i8\times10} = 0 \quad \Xi_{i8\times11} = \sqrt{d_2}N_7$

$\Xi_{i8\times12} = \sqrt{d}M_7 \quad \Xi_{i9\times9} = -\gamma^2 I \quad \Xi_{i9\times10} = -D_i^T D_{fi}^T \quad \Xi_{i9\times11} = \Xi_{i9\times12} = 0 \quad \Xi_{i10\times10} = -I$

$\Xi_{i10\times11} = \Xi_{i10\times12} = 0 \quad \Xi_{i11\times11} = -Z_3 \quad \Xi_{i11\times12} = 0 \quad \Xi_{i12\times12} = -Z_3$

$\alpha_1 = \left[\bar{M}_{1i}^T W_i, \bar{M}_{1i}^T V_i, 0, 0, 0, 0, 0, 0, 0, 0, \bar{M}_{2i}^T, 0, 0\right]^T$

$\alpha_2 = \left[\bar{N}_{1i}, 0, \bar{N}_{2i}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right] \quad RE_i = 0\left(R \in \mathbb{R}^{(n-r)\times n}\right)$

$\bar{M}_{1i} = \begin{bmatrix} \tilde{M}_{1i} \\ B_{fi}\tilde{M}_{2i} \end{bmatrix} \quad \bar{M}_{2i} = \tilde{M}_{3i} - D_{fi}\tilde{M}_{2i} \quad \bar{N}_{1i} = \begin{bmatrix} \tilde{N}_{1i} & 0 \end{bmatrix} \quad \bar{N}_{2i} = \begin{bmatrix} \tilde{N}_{2i} & 0 \end{bmatrix}$

Let

$W_i^T = \begin{bmatrix} W_{1i} & \mathcal{X}_i \\ W_{2i} & \mathcal{X}_i \end{bmatrix} \quad V_i^T = \begin{bmatrix} V_{1i} & \mathcal{X}_i \\ V_{2i} & \mathcal{X}_i \end{bmatrix} \quad \hat{E} = \begin{bmatrix} E_i & 0 \\ 0 & E_i \end{bmatrix} \quad \hat{R} = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$

$\bar{Q}_3 = \sum_{j=1}^{s}\pi_{ij}Q_{3j} = \begin{bmatrix} \bar{Q}_{11}^3 & \bar{Q}_{12}^3 \\ * & \bar{Q}_{22}^3 \end{bmatrix} \quad \bar{S} = \sum_{j=1}^{s}\pi_{ij}S_j = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} \\ * & \bar{S}_{22} \end{bmatrix} \quad \bar{P} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ * & \bar{P}_{22} \end{bmatrix}$

$T_i = \begin{bmatrix} T_{i1} & T_{i2} \\ T_{i3} & T_{i4} \end{bmatrix} \quad P_i = \begin{bmatrix} P_{i11} & P_{i12} \\ * & P_{i22} \end{bmatrix} \quad Q = \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix} \quad S_i = \begin{bmatrix} S_{i1} & S_{i2} \\ * & S_{i3} \end{bmatrix}$

$N_j = \begin{bmatrix} N_{j1} & 0 \\ 0 & N_{j2} \end{bmatrix} \quad M_j = \begin{bmatrix} M_{j1} & 0 \\ 0 & M_{j2} \end{bmatrix} \quad (j = 1, 2, \cdots 7) \quad S = \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix}$

$Q_{k_1} = \begin{bmatrix} Q_{11}^{k_1} & Q_{12}^{k_1} \\ * & Q_{22}^{k_1} \end{bmatrix} (k_1 = 1, 2, 3) \quad Z_{k_2} = \begin{bmatrix} Z_{11}^{k_2} & Z_{12}^{k_2} \\ * & Z_{22}^{k_2} \end{bmatrix} (k_2 = 1, 2, 3) \quad Q_{3i} = \begin{bmatrix} Q_{11}^{3i} & Q_{12}^{3i} \\ * & Q_{22}^{3i} \end{bmatrix}_{\forall i \in \mathcal{S}}$

Then,

$\Xi_{il\times k} = \begin{bmatrix} \Xi_{il\times k}^1 & \Xi_{il\times k}^2 \\ * & \Xi_{il\times k}^3 \end{bmatrix} \quad l, k = 1, 2, \cdots 12$

$\Xi_{i1\times1}^1 = W_{1i}A_i + (W_{1i}A_i)^T + B_{Fi}C_i + (B_{Fi}C_i)^T + E_i^T \bar{P}_{11}E_i + Q_{11}^1 + Q_{11}^2 + d_1 Q_{11}^3 + Q_{11}^{3i} + d_2 Q_{11}$
$\quad + dS_{i1} + \frac{1}{2}\left(d_2^2 - d_1^2\right)S_{11} - \frac{4}{d_1}E_i^T Z_{11}^1 E_i + N_{11}E_i + (N_{11}E_i)^T$

$\Xi_{i1\times1}^2 = A_{Fi} + A_i^T W_{2i}^T + C_i^T B_{Fi}^T + E_i^T \bar{P}_{12}E_i + Q_{12}^1 + Q_{12}^2 + d_1 Q_{12}^3 + Q_{12}^{3i} + d_2 Q_{12} + dS_{i2}$
$\quad + \frac{1}{2}\left(d_2^2 - d_1^2\right)S_{12} - \frac{4}{d_1}E_i^T Z_{12}^1 E_i$

$\Xi_{i1\times1}^3 = A_{Fi} + A_{Fi}^T + E_i^T \bar{P}_{22}E_i + Q_{22}^1 + Q_{22}^2 + d_1 Q_{22}^3 + Q_{22}^{3i} + d_2 Q_{22} + dS_{i3} + \frac{1}{2}\left(d_2^2 - d_1^2\right)S_{22}$
$\quad - \frac{4}{d_1}E_i^T Z_{22}^1 E_i + N_{12}E_i + (N_{12}E_i)^T$

$$\Xi_{i1\times2} = \begin{bmatrix} E_i^T P_{i11} + T_{i1}R - W_{1i} + A_i^T V_{1i}^T + C_i^T B_{Fi}^T & E_i^T P_{i12} + T_{i2}R - \mathcal{X}_i + A_i^T V_{2i}^T + C_i^T B_{Fi}^T \\ E_i^T P_{i12} + T_{i3}R - W_{2i} + A_{Fi}^T & E_i^T P_{i22} + T_{i4}R - \mathcal{X}_i + A_{Fi}^T \end{bmatrix}$$

$$\Xi_{i1\times3} = \begin{bmatrix} W_{1i}A_{di} + B_{Fi}C_{di} - N_{11}E_i + E_i^T N_{21}^T + M_{11}E_i & 0 \\ W_{2i}A_{di} + B_{Fi}C_{di} & -N_{12}E_i + E_i^T N_{22}^T + M_{12}E_i \end{bmatrix}$$

$$\begin{bmatrix} \Xi_{i1\times4} & \Xi_{i1\times5} \end{bmatrix} = \begin{bmatrix} -\frac{2}{d_1}E_i^T Z_{11}^1 E_i + E_i^T N_{31}^T, -\frac{2}{d_1}E_i^T Z_{21}^1 E_i, E_i^T N_{41}^T - M_{11}E_i, 0 \\ -\frac{2}{d_1}E_i^T \left(Z_{12}^1\right)^T E_i, -\frac{2}{d_1}E_i^T Z_{22}^1 E_i + E_i^T N_{32}^T, 0, E_i^T N_{42}^T - M_{12}E_i \end{bmatrix}$$

$$\begin{bmatrix} \Xi_{i1\times6} & \Xi_{i1\times7} \end{bmatrix} = \begin{bmatrix} \frac{6}{d_1^2}E_i^T Z_{11}^1 E_i + E_i^T N_{51}^T, \frac{6}{d_1^2}E_i^T Z_{12}^1 E_i, E_i^T N_{61}^T, 0 \\ \frac{6}{d_1^2}E_i^T \left(Z_{12}^1\right)^T E_i, \frac{6}{d_1^2}E_i^T Z_{22}^1 E_i + E_i^T N_{52}^T, 0, E_i^T N_{62}^T \end{bmatrix}$$

$$\begin{bmatrix} \Xi_{i1\times8} & \Xi_{i1\times9} & \Xi_{i1\times10} \end{bmatrix} = \begin{bmatrix} E_i^T N_{71}^T, 0, W_{1i}B_i + B_{Fi}D_i, L_i^T - C_i^T D_{Fi}^T \\ 0, E_i^T N_{72}^T, W_{2i}B_i + B_{Fi}D_i, -C_{Fi}^T \end{bmatrix}$$

$$\begin{bmatrix} \Xi_{i1\times11} & \Xi_{i1\times12} \end{bmatrix} = \begin{bmatrix} \sqrt{d_2}N_{11} & 0 & \sqrt{d}M_{11} & 0 \\ 0 & \sqrt{d_2}N_{12} & 0 & \sqrt{d}M_{12} \end{bmatrix}$$

$$\Xi_{i2\times2} = \begin{bmatrix} -V_{1i} - V_{1i}^T + d_1 Z_{11}^1 + d_2 Z_{11}^3 + dZ_{11}^2, -V_{2i}^T - \mathcal{X}_i + d_1 Z_{12}^1 + d_2 Z_{12}^3 + dZ_{12}^2 \\ * \quad -\mathcal{X}_i - \mathcal{X}_i^T + d_1 Z_{22}^1 + d_2 Z_{22}^3 + dZ_{22}^2 \end{bmatrix}$$

$$\Xi_{i2\times3}=\begin{bmatrix} V_{1i}A_{di}+B_{Fi}C_{di} & 0 \\ V_{2i}A_{di}+B_{Fi}C_{di} & 0 \end{bmatrix} \qquad \Xi_{i2\times9}=\begin{bmatrix} V_{1i}B_{i}+B_{Fi}D_{i} \\ V_{2i}B_{i}+B_{Fi}D_{i} \end{bmatrix}$$

$$\Xi^1_{i3\times3}=-\left(1-\mu\right)Q^{3i}_{11}-\tfrac{8}{d}E^T_iZ^2_{11}E_i-N_{21}E_i-\left(N_{21}E_i\right)^T+M_{21}E_i+\left(M_{21}E_i\right)^T$$

$$\Xi^2_{i3\times3}=-\left(1-\mu\right)Q^{3i}_{12}-\tfrac{8}{d}E^T_iZ^2_{12}E_i$$

$$\Xi^3_{i3\times3}=-\left(1-\mu\right)Q^{3i}_{22}-\tfrac{8}{d}E^T_iZ^2_{22}E_i-N_{22}E_i-\left(N_{22}E_i\right)^T+M_{22}E_i+\left(M_{22}E_i\right)^T$$

$$\Xi_{i3\times4}=\begin{bmatrix} -\tfrac{2}{d}E^T_iZ^2_{11}E_i-E^T_iN^T_{31}+E^T_iM^T_{31} & -\tfrac{2}{d}E^T_iZ^2_{12}E_i \\ -\tfrac{2}{d}E^T_i\left(Z^2_{12}\right)^TE_i & -\tfrac{2}{d}E^T_iZ^2_{22}E_i-E^T_iN^T_{32}+E^T_iM^T_{32} \end{bmatrix}$$

$$\Xi_{i3\times5}=\begin{bmatrix} -\tfrac{2}{d}E^T_iZ^2_{11}E_i-E^T_iN^T_{41}-M_{21}E_i+E^T_iM^T_{41} & -\tfrac{2}{d}E^T_iZ^2_{12}E_i \\ -\tfrac{2}{d}E^T_i\left(Z^2_{12}\right)^TE_i & -\tfrac{2}{d}E^T_iZ^2_{22}E_i-E^T_iN^T_{42}-M_{22}E_i+E^T_iM^T_{42} \end{bmatrix}$$

$$\begin{bmatrix} \Xi_{i3\times6} & \Xi_{i3\times7} \end{bmatrix}=\begin{bmatrix} -E^T_iN^T_{51}+E^T_iM^T_{51},0,\tfrac{6}{d^2}E^T_iZ^2_{11}E_i-E^T_iN^T_{61}+E^T_iM^T_{61},\tfrac{6}{d^2}E^T_iZ^2_{12}E_i \\ 0,-E^T_iN^T_{52}+E^T_iM^T_{52},\tfrac{6}{d^2}E^T_i\left(Z^2_{12}\right)^TE_i,\tfrac{6}{d^2}E^T_iZ^2_{22}E_i-E^T_iN^T_{62}+E^T_iM^T_{62} \end{bmatrix}$$

$$\begin{bmatrix} \Xi_{i3\times8} & \Xi_{i3\times9} & \Xi_{i3\times10} \end{bmatrix}=\begin{bmatrix} \tfrac{6}{d^2}E^T_iZ^2_{11}E_i-E^T_iN^T_{71}+E^T_iM^T_{71},\tfrac{6}{d^2}E^T_iZ^2_{12}E_i,0,L^T_{di}-C^T_{di}D^T_{Fi} \\ \tfrac{6}{d^2}E^T_i\left(Z^2_{12}\right)^TE_i,\tfrac{6}{d^2}E^T_iZ^2_{22}E_i-E^T_iN^T_{72}+E^T_iM^T_{72},0,0 \end{bmatrix}$$

$$\begin{bmatrix} \Xi_{i3\times11} & \Xi_{i3\times12} \end{bmatrix}=\begin{bmatrix} \sqrt{d_2}N_{21} & 0 & \sqrt{d}M_{21} & 0 \\ 0 & \sqrt{d_2}N_{22} & 0 & \sqrt{d}M_{22} \end{bmatrix}$$

$$\Xi_{i4\times4}=\begin{bmatrix} -Q^1_{11}-\tfrac{4}{d_1}E^T_iZ^1_{11}E_i-\tfrac{4}{d}E^T_iZ^2_{11}E_i & -Q^1_{12}-\tfrac{4}{d_1}E^T_iZ^1_{12}E_i-\tfrac{4}{d}E^T_iZ^2_{12}E_i \\ * & -Q^1_{22}-\tfrac{4}{d_1}E^T_iZ^1_{22}E_i-\tfrac{4}{d}E^T_iZ^2_{22}E_i \end{bmatrix}$$

$$\begin{bmatrix} \Xi_{i4\times5} & \Xi_{i4\times6} & \Xi_{i4\times7} \end{bmatrix}=\begin{bmatrix} -M_{31}E_i,0,\tfrac{6}{d_1^2}E^T_iZ^1_{11}E_i,\tfrac{6}{d_1^2}E^T_iZ^1_{12}E_i,\tfrac{6}{d_1^2}E^T_iZ^2_{11}E_i,\tfrac{6}{d_1^2}E^T_iZ^2_{12}E_i \\ 0,-M_{32}E_i,\tfrac{6}{d_1^2}E^T_i\left(Z^1_{12}\right)^TE_i,\tfrac{6}{d_1^2}E^T_iZ^1_{22}E_i,\tfrac{6}{d_1^2}E^T_i\left(Z^2_{12}\right)^TE_i,\tfrac{6}{d_1^2}E^T_iZ^2_{22}E_i \end{bmatrix}$$

$$\begin{bmatrix} \Xi_{i4\times11} & \Xi_{i4\times12} \end{bmatrix}=\begin{bmatrix} \sqrt{d_2}N_{31} & 0 & \sqrt{d}M_{31} & 0 \\ 0 & \sqrt{d_2}N_{32} & 0 & \sqrt{d}M_{32} \end{bmatrix}$$

$$\Xi_{i5\times5}=\begin{bmatrix} -Q^2_{11}-\tfrac{4}{d}E^T_iZ^2_{11}E_i-M_{41}E_i-\left(M_{41}E_i\right)^T & -Q^2_{12}-\tfrac{4}{d}E^T_iZ^2_{12}E_i \\ * & -Q^2_{22}-\tfrac{4}{d}E^T_iZ^2_{22}E_i-M_{42}E_i-\left(M_{42}E_i\right)^T \end{bmatrix}$$

$$\begin{bmatrix} \Xi_{i5\times6} & \Xi_{i5\times7} & \Xi_{i5\times8} \end{bmatrix}=\begin{bmatrix} -E^T_iM^T_{51},0,-E^T_iM^T_{61},0,\tfrac{6}{d^2}E^T_iZ^2_{11}E_i-E^T_iM^T_{71},\tfrac{6}{d^2}E^T_iZ^2_{12}E_i \\ 0,-E^T_iM^T_{52},0,-E^T_iM^T_{62},\tfrac{6}{d^2}E^T_i\left(Z^2_{12}\right)^TE_i,\tfrac{6}{d^2}E^T_iZ^2_{22}E_i-E^T_iM^T_{72} \end{bmatrix}$$

$$\begin{bmatrix} \Xi_{i5\times11} & \Xi_{i5\times12} \end{bmatrix}=\begin{bmatrix} \sqrt{d_2}N_{41} & 0 & \sqrt{d}M_{41} & 0 \\ 0 & \sqrt{d_2}N_{42} & 0 & \sqrt{d}M_{42} \end{bmatrix}$$

$$\Xi_{i6\times6}=\begin{bmatrix} -\tfrac{1}{d_1}Q^3_{11}-\tfrac{12}{d_1^3}E^T_iZ^1_{11}E_i & -\tfrac{1}{d_1}Q^3_{12}-\tfrac{12}{d_1^3}E^T_iZ^1_{12}E_i \\ * & -\tfrac{1}{d_1}Q^3_{22}-\tfrac{12}{d_1^3}E^T_iZ^1_{22}E_i \end{bmatrix}$$

$$\begin{bmatrix} \Xi_{i6\times11} & \Xi_{i6\times12} \end{bmatrix}=\begin{bmatrix} \sqrt{d_2}N_{51} & 0 & \sqrt{d}M_{51} & 0 \\ 0 & \sqrt{d_2}N_{52} & 0 & \sqrt{d}M_{52} \end{bmatrix}$$

$$\Xi_{i7\times7}=\Xi_{i8\times8}=\begin{bmatrix} -\tfrac{1}{d}S_{i1}-\tfrac{12}{d^3}E^T_iZ^2_{11}E_i & -\tfrac{1}{d}S_{i2}-\tfrac{12}{d^3}E^T_iZ^2_{12}E_i \\ * & -\tfrac{1}{d}S_{i3}-\tfrac{12}{d^3}E^T_iZ^2_{22}E_i \end{bmatrix}$$

$$\begin{bmatrix} \Xi_{i7\times11} & \Xi_{i7\times12} \end{bmatrix}=\begin{bmatrix} \sqrt{d_2}N_{61} & 0 & \sqrt{d}M_{61} & 0 \\ 0 & \sqrt{d_2}N_{62} & 0 & \sqrt{d}M_{62} \end{bmatrix}$$

$$\begin{bmatrix} \Xi_{i8\times11} & \Xi_{i8\times12} \end{bmatrix}=\begin{bmatrix} \sqrt{d_2}N_{71} & 0 & \sqrt{d}M_{71} & 0 \\ 0 & \sqrt{d_2}N_{72} & 0 & \sqrt{d}M_{72} \end{bmatrix}$$

$$\Xi_{i9\times9}=-\gamma^2I \qquad \Xi_{i9\times10}=-D^T_iD^T_{Fi} \qquad \Xi_{i10\times10}=-I \qquad \Xi_{i11\times11}=\Xi_{i12\times12}=\begin{bmatrix} -Z^3_{11} & -Z^3_{12} \\ * & -Z^3_{22} \end{bmatrix}$$

Now we can find that there exists a mode-dependent filter (7) such that the desired mode-dependent filter realization is given by

$$A_{fi}=\mathcal{X}_i^{-1}A_{Fi} \qquad B_{fi}=\mathcal{X}_i^{-1}B_{Fi} \qquad C_{fi}=C_{Fi} \qquad D_{fi}=D_{Fi} \tag{3.38}$$

Proof. Let the matrix $\Xi_i\left(t\right)$ is expressed as the uncertainty form of matrix Ξ_i. Now, we consider the matrix

$$\Psi_i(t) = \begin{bmatrix} I & \hat{A}_i^T & 0_{n\times 10n} \\ 0_{10n\times n} & \alpha_3 & I_{10n\times 10n} \end{bmatrix}_{11n\times 12n} \tag{3.39}$$

where $\alpha_3 = \begin{pmatrix} \hat{A}_{di}(t) & 0_{n\times 5n} & \hat{B}_i & 0_{n\times 3n} \end{pmatrix}^T$

Pre-multiplying and post-multiplying $\Xi_i(t)$ by the matrix $\Psi_i(t)$ and its transposition, respectively. Then, we can find that for $\forall i \in \mathcal{S}$ such that

$$\Phi_i = \Psi_i(t)\,\Xi_i(t)\,\Psi_i^T(t) \tag{3.40}$$

hold. So, if $\Xi_i(t) < 0$ implies $\Phi_i < 0$. Furthermore,

$$\Xi_i(t) = \Xi_i + \alpha_1 F_i(t)\,\alpha_2^T + \alpha_2 F_i^T(t)\,\alpha_1^T \tag{3.41}$$

Based on the Lemma 2.3 and Lemma 2.5, we can obtain that $\Xi_i(t) < 0$ is equivalent to the (53) holds. Thus, connecting with (51)-(53), we can obtain (17)-(19). At the same time, the parameters of the desired mode-dependent filter in (7) are written as (54). This completes the proof. □

Remark 3.1 From the Proof of the **Theorem 3.2**, we can find that the matrix (55) plays a very important role in our discussion. As well known, if we want to obtain the optimal filtering gain matrices, the key step is how to divide the nonlinear terms, so, this is why we introduce the matrix (55), which makes our proof more easily.

Remark 3.2 It should be point out that the likelihood to obtain the complete knowledge on the transition probabilities is difficult and the cost is probably high. Thus, in order to more accurately describe the practical problem, we need to further study more general Markovian jump systems with incomplete transition descriptions. Based on this consideration, the idea and method in this paper could be extended to cope with the singular Markovian jump systems with partly unknown transition rates.

Remark 3.3 It is worth mentioning that in most of the works about Markovian jump systems with mode-dependent delay, the delay mode is assumed to be the same as the system matrices mode. But this is natural in modeling of real systems, the delay mode may not be the same as that for jumps in other system parameters, that is, variations of delay usually depend on phenomena which may not cause abrupt changes in the other systems parameters. Thus, discussing of the work about singular Markovian jump systems with different system modes is not only theoretically interesting and challenging, but also very important in practical applications. So, this problem will be considered in our future work.

4 Examples

In this section, we shall give an illustrative example to demonstrate the effectiveness of the proposed method. Firstly we consider the transition rate matrix Π with two vertices and singular matrix $E_i, i \in \mathcal{S}$:

$$\Pi = \begin{bmatrix} -0.6 & 0.6 \\ 0.8 & -0.8 \end{bmatrix} \qquad E_i = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Secondly, we introduce the following parameters into the error system (8):

$$A_1 = \begin{bmatrix} -0.4 & -1.2 \\ 0 & -0.3 \end{bmatrix} \qquad A_{d1} = \begin{bmatrix} 0.5 & -0.3 \\ 0.1 & -0.8 \end{bmatrix} \qquad B_1 = \begin{bmatrix} 0.2 \\ -0.3 \end{bmatrix} \qquad C_1 = \begin{bmatrix} 0.6 & 0 \end{bmatrix}$$

$$C_{d1} = \begin{bmatrix} 0.6 & -0.6 \end{bmatrix} \quad D_1 = 0.2 \quad L_1 = \begin{bmatrix} 0.3 & 0.9 \end{bmatrix} \quad L_{d1} = \begin{bmatrix} -0.3 & 0.1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -0.1 & 0.7 \\ -0.4 & -1.3 \end{bmatrix} \quad A_{d2} = \begin{bmatrix} -0.6 & 0.5 \\ 0.2 & -0.8 \end{bmatrix} \quad B_2 = \begin{bmatrix} -0.1 \\ 0.5 \end{bmatrix} \quad C_2 = \begin{bmatrix} -0.3 & -0.1 \end{bmatrix}$$

$$C_{d2} = \begin{bmatrix} 0.8 & -0.1 \end{bmatrix} \quad D_2 = 0.3 \quad L_2 = \begin{bmatrix} 0.3 & -0.3 \end{bmatrix} \quad L_{d2} = \begin{bmatrix} -0.5 & 0.2 \end{bmatrix}$$

$$\tilde{M}_{11} = \tilde{M}_{12} = \begin{bmatrix} 0.5 \\ -0.2 \end{bmatrix} \quad \tilde{M}_{21} = \tilde{M}_{22} = \tilde{M}_{31} = \tilde{M}_{32} = 0.1$$

$$\tilde{N}_{11} = \tilde{N}_{12} = \begin{bmatrix} 0.1 & -0.1 \end{bmatrix} \quad \tilde{N}_{21} = \tilde{N}_{22} = \begin{bmatrix} 0.2 & 0.4 \end{bmatrix}$$

and matrix R, which satisfying $RE = 0$

$$R = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Let $d_1 = 0.1$, $d_2 = 0.5$, $\mu = 0.5$. Solving LMIs (51)-(53) in Theorem 3.2, we obtain the minimum feasible $\gamma^* = 0.2143$, and by solving equation (54), we obtain the parameters of the desired filter (7) as follows:

$$A_{f1} = \begin{bmatrix} -0.0038 & -0.1517 \\ -0.0163 & -0.1161 \end{bmatrix} \quad B_{f1} = \begin{bmatrix} 0.0017 \\ -0.0142 \end{bmatrix} \quad C_{f1} = \begin{bmatrix} 0.2370 & 0.6125 \end{bmatrix} \quad D_{f1} = -0.0167 \tag{4.1}$$

$$A_{f2} = \begin{bmatrix} -0.0243 & -0.0069 \\ -0.0144 & -0.0081 \end{bmatrix} \quad B_{f2} = \begin{bmatrix} -0.0092 \\ -0.0180 \end{bmatrix} \quad C_{f2} = \begin{bmatrix} -0.0331 & -0.0101 \end{bmatrix} \quad D_{f2} = -0.1143 \tag{4.2}$$

Remark 4.1 Based on the Theorem 3.2, we can get the corresponding filter parameters for different values of γ and d_2, meantime, Table 1 shows the minimum performance γ^* in terms of the feasibility of (51)-(53) for different d_2. Obviously, form the Table 1, we can discover that the more larger upper bound of time delay $d(t)$, the more poor performance of a system, that is, it will worse disturbance attenuation performance level γ^*.

In order to show the effectiveness of the filer system (7) with parameters (58) and (59), let initial conditions $x_0 = \begin{bmatrix} 0.1 & -0.1 \end{bmatrix}^T$, $x_{f0} = \begin{bmatrix} 0.2 & -0.5 \end{bmatrix}^T$, the simulations of system mode r_r, system state $x(t)$, filter state $x_f(t)$ and error estimation signal $\hat{z}(t)$ are demonstrated in Figures 1, Figures 2, Figures 3 and Figures 4, respectively, which show the validity of the presented results.

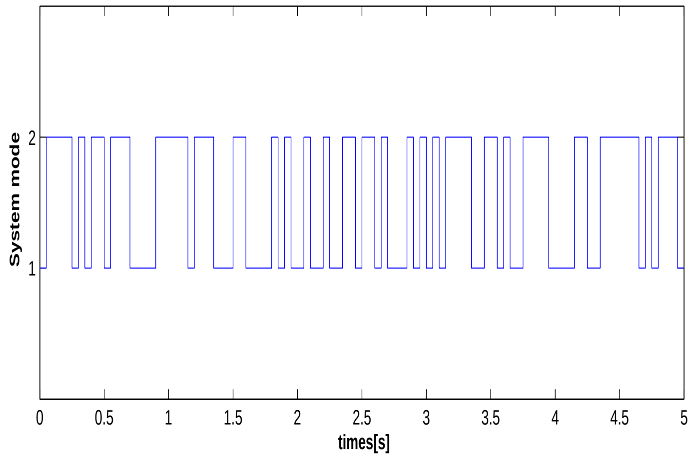

Fig. 1. system jumping mode r_t

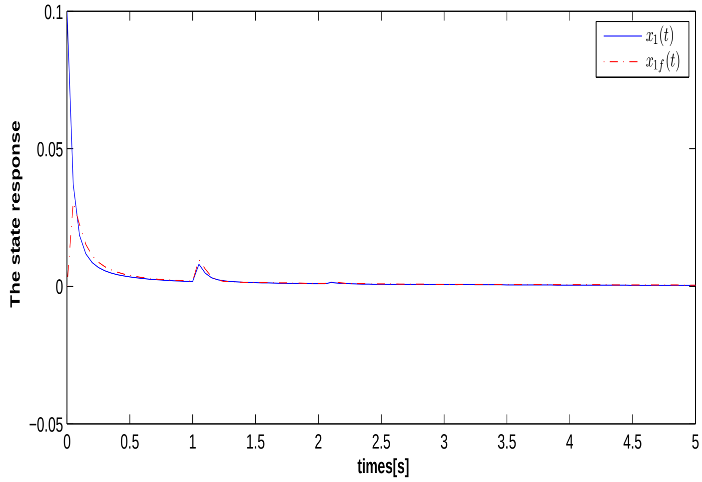

Fig. 2. The state responses of $x_1\left(t\right)$ and $x_{f1}\left(t\right)$

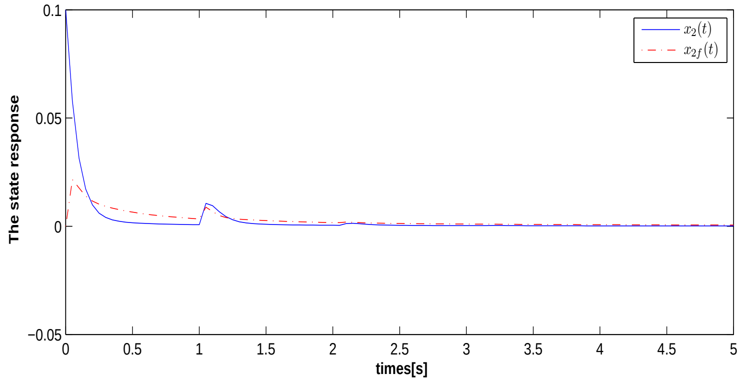

Fig. 3. The state responses of $x_2(t)$ and $x_{f2}(t)$

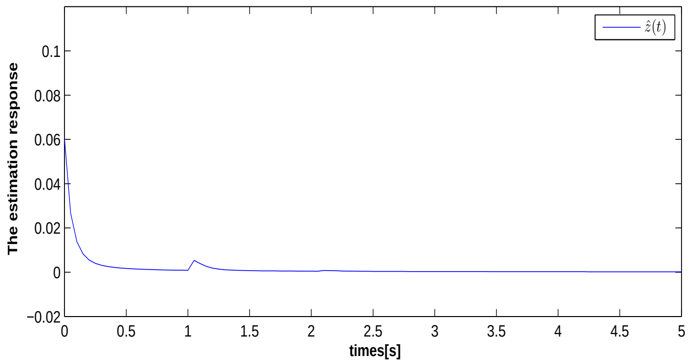

Fig. 4. The estimation error response of $\hat{z}(t)$

Table 1. Calculated the minimum γ^* with different d_2

d_2	0.5	0.7	0.9	1.2	1.5	1.6	1.8	2.0
γ^*	0.4143	0.4145	0.4148	0.5349	0.5517	0.6152	0.6654	0.9155

5 Conclusion

In this paper, the exponential H_∞ filtering design problem is investigated for a class of uncertain singular Markovian jump systems with time-varying delay. A mode-dependent desired filter has been designed, which ensures the filtering error system is not only exponentially mean-square stable, but also satisfies a prescribed H_∞-norm level for all admissible uncertainties. The obtained result is presented in terms of LMLs, which depends on the bounds of the time-varying delays and can be obtained easily by using Matlab LMI Toolbox. Finally, a numerical example is employed to demonstrate the feasibility of the proposed approach.

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Competing Interests

Authors have declared that no competing interests exist.

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